1 Example

Exercise. *Suppose $T(1) = 3$ and $T(n) = 3T(n/2) + n$. How would you find $T(8)$? The point of this exercise is the process.*

This is the same approach that's used to prove the Master Theorem.

2 Master Theorem

Start with a recurrence $T(n) = aT(n/b) + cn^k$ (supposing that $T(p_0) = q_0$ for constants p_0 and q_0) and expand:

$$\begin{aligned} T(n) &= aT(n/b) + cn^k \\ &= a\left[aT(n/b^2) + c\left(\frac{n}{b}\right)^k\right] + cn^k = a^2T(n/b^2) + cn^k\left(1 + \frac{a}{b^k}\right) \\ &\vdots \\ &= a^sT(n/b^s) + cn^k\left[\left(\frac{a}{b^k}\right)^s + \left(\frac{a}{b^k}\right)^{s-1} + \ldots + \frac{a}{b^k} + 1\right] \end{aligned}$$

We stop expanding when we reach the base case, when $\frac{n}{b^s} = p_0$. This occurs after $s \approx \log_b\left(\frac{n}{p_0}\right) = \log_b n +$ constant iterations. Notice that the expression is split into two terms. The asymptotic form of $T(n)$ is just a competition between these two terms to see which one dominates.

The second term has a geometric sum: using the formula for a geometric sum gives:

$$T(n) = a^s q_0 + cn^k\left[\frac{1 - \left(\frac{a}{b^k}\right)^{s+1}}{1 - \frac{a}{b^k}}\right]$$

Exercise. *Use the above expansion to derive the case of the Master Theorem for $a < b^k$.*

Exercise. *Now derive the Master Theorem for* $a > b^k$*.*

Exercise. *Derive the Master Theorem for* $a = b^k$*.*

Qualitatively, if $a > b^k$, the bottleneck of the recurrence is the number of recursive calls we have to make. Otherwise, it's the extra work done *during* each call (i.e. the cn^k term) that dominates the runtime.