1 Example

Exercise. *Suppose $T(1) = 3$ and $T(n) = 3T(n/2) + n$. How would you find $T(8)$? The point of this exercise is the process.*

Solution.
Expand and substitute using the formula for the recurrence:

$$\begin{aligned} T(8) &= 3T(4) + 8 \\ &= 3\left[3T(2) + 4\right] + 8 = 9T(2) + 20 \\ &= 9\left[3T(1) + 2\right] + 20 = 27T(1) + 38 = 119 \end{aligned}$$

This is the same approach that's used to prove the Master Theorem.

2 Master Theorem

Start with a recurrence $T(n) = aT(n/b) + cn^k$ (supposing that $T(p_0) = q_0$ for constants p_0 and q_0) and expand:

$$\begin{aligned} T(n) &= aT(n/b) + cn^k \\ &= a\left[aT(n/b^2) + c\left(\frac{n}{b}\right)^k\right] + cn^k = a^2T(n/b^2) + cn^k\left(1 + \frac{a}{b^k}\right) \\ &\vdots \\ &= a^sT(n/b^s) + cn^k\left[\left(\frac{a}{b^k}\right)^s + \left(\frac{a}{b^k}\right)^{s-1} + \ldots + \frac{a}{b^k} + 1\right] \end{aligned}$$

We stop expanding when we reach the base case, when $\frac{n}{b^s} = p_0$. This occurs after $s \approx \log_b\left(\frac{n}{p_0}\right) = \log_b n +$ constant iterations. Notice that the expression is split into two terms. The asymptotic form of $T(n)$ is just a competition between these two terms to see which one dominates.

The second term has a geometric sum: using the formula for a geometric sum gives:

$$T(n) = a^s q_0 + cn^k\left[\frac{1 - \left(\frac{a}{b^k}\right)^{s+1}}{1 - \frac{a}{b^k}}\right]$$

Exercise. *Use the above expansion to derive the case of the Master Theorem for $a < b^k$.*

Solution.
Here, $\frac{a}{b^k} < 1$, and as n (and therefore s) grows large the sum of the above geometric series is dominated by the constant term $\frac{1}{1-\frac{a}{b^k}} = \Theta(1)$. So $T(n) = \Theta(a^s) + \Theta(n^k)$. Using our expression for s:

$$a^s = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a}) = o(n^k)$$

since $a < b^k$ means that $\log_b a < k$. We therefore get that $T(n) = o\left(n^k\right) + \Theta(n^k) = \Theta(n^k)$.

Exercise. *Now derive the Master Theorem for $a > b^k$.*

Solution.
Proceeding like the previous case, the geometric sum is now dominated by the:

$$\frac{\left(\frac{a}{b^k}\right)^{s+1}}{\frac{a}{b^k} - 1} = \Theta\left(\left(\frac{a}{b^k}\right)^s\right)$$

term. Then the second term of $T(n)$ is:

$$cn^k \cdot \Theta\left(\left(\frac{a}{b^k}\right)^{\log_b n}\right) = cn^k \cdot \Theta\left(\frac{n^{\log_b a}}{n^k}\right) = \Theta\left(n^{\log_b a}\right)$$

This along with the result from the previous exercise that $a^s = \Theta\left(n^{\log_b a}\right)$ gives that $T(n) = \Theta\left(n^{\log_b a}\right)$.

Exercise. *Derive the Master Theorem for $a = b^k$.*

Solution.
Every term in the geometric series is now 1. There are $s+1$ terms, so the second term of $T(n)$ becomes:

$$cn^k(s+1) = \Theta\left(n^k \log_b n\right) = \Theta\left(n^k \log n\right)$$

The first term of $T(n)$ is $\Theta\left(n^{\log_b a}\right) = \Theta\left(n^k\right)$ so the second term dominates and $T(n) = \Theta\left(n^k \log n\right)$.

Qualitatively, if $a > b^k$, the bottleneck of the recurrence is the number of recursive calls we have to make. Otherwise, it's the extra work done *during* each call (i.e. the cn^k term) that dominates the runtime.