7.1 The Origin of Computer Science

Alan Mathison Turing (1912–1954)

“On Computable Numbers, with an Application to the Entscheidungsproblem” 1936

- 1936: Founded computer science as graduate student
- 1938–1945: WW II hero — breaking of German codes
- 1948: Almost qualified for UK Olympic team as marathon runner
- 1950: Seminal paper on AI — proposed Turing test
- 1952: Prosecuted for homosexuality, chose chemical castration over prison
- 1954: Suicide
- 1966: Turing Award introduced as highest prize in computer science
- 2009: Public apology by British government
- 2013: Pardon by Queen of England

[Dates from `http://en.wikipedia.org/wiki/Alan_Turing`]

Q: What Problem Was Turing Trying to Solve?

David Hilbert

"Mathematical Problems" 1900

Can mathematics be fully axiomatized and automated?

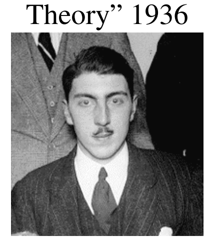
Kurt Gödel

"On Formally Undecidable Propositions . . . " 1931

Mathematics cannot be fully axiomatized: some true statements will be unprovable.

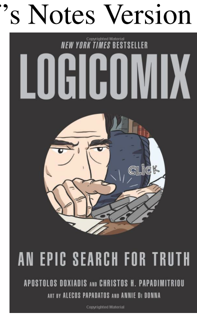
Alonzo Church

"An Unsolvable Problem of Elementary Number Theory" 1936

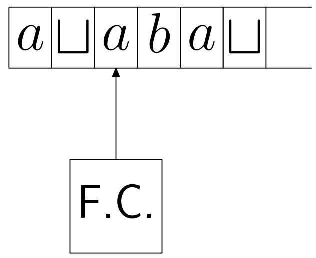


Independently of Turing: mathematics cannot be automated — cannot algorithmically distinguish provable from disprovable statements.

The Cliff's Notes Version of History



7.2 The Basic Turing Machine



- Head can both read and write, and move in both directions
- Tape has unbounded length
- $\sqcup$ is the blank symbol. All but a finite number of tape squares are blank.
- F.C. = finite-state control.

Def: A (deterministic) Turing Machine (TM) is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{halt})$, where:

- Q is a finite set of states, containing a *start state* q_0 and a *halt state* q_{halt}.
- Σ is the input alphabet
- Γ is the tape alphabet, containing Σ and $\sqcup \in \Gamma - \Sigma$.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function

Interpretation:

- $\delta(q, \sigma) = (q', \sigma', R)$
  - Rewrite σ as σ' in current cell
  - Switch from state q to state q'
  - And move right
- $\delta(q, \sigma) = (q', \sigma', L)$
  - Same, but move left
  - *Unless* at left end of tape, in which case stay put

7.3 An Example

A Turing machine to determine whether a string is an even-length palindrome over alphabet $\Sigma = \{a, b\}$:

7.4 Formalizing Computation of TMs

- A configuration of a TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{halt})$ is denoted uqv, where $q \in Q$, $u, v \in \Gamma^*$.
  - Tape contents = uv followed by all blanks
  - State = q
  - Head on first symbol of v.
  - *Equivalent* to uqv', where $v' = v\sqcup$.
- Start configuration = q_0w, where w is input.
- One step of computation (denoted $C \Rightarrow_M C'$): for $C = uq\sigma v$ with $q \in Q \setminus \{q_{halt}\}$, $u, v \in \Gamma^*$, $\sigma \in \Gamma$,
  - If $\delta(q, \sigma) = (q', \sigma', R)$, then $C' = u\sigma' q' v$.
  - If $\delta(q, \sigma) = (q', \sigma', L)$, and $u = u'\tau$ for $u' \in \Gamma^*$ and $\tau \in Gamma$, then $C' = u'q'\tau\sigma' v$.
  - If $\delta(q, \sigma) = (q', \sigma', L)$, and $u = \varepsilon$, then $C' = q'\sigma' v$.
- If $q = q_{halt}$, computation halts ($C' = C$) and the *output* is the contents of the tape to the left of the first blank symbol.

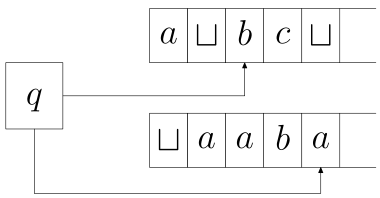
Def: TM $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{halt})$ *solves* computational problem $f : \Sigma^* \Rightarrow 2^{(\Gamma \setminus \{\sqcup\})^*}$ if for every $w \in \Sigma^*$, there is a sequence $C_0, \ldots, C_t$ of configurations of M such that:

1. $C_0 = q_0 w$
2. $C_{i-1} \Rightarrow_M C_i$ for $i = 1, \ldots, t$
3. In C_t, M is state q_{halt} and the contents of the tape to the left of the first blank symbol is an element of $f(w)$.

Running time: Defined analogously to Word-RAM, e.g. $T(n)$ = maximum number of steps taken by M on all inputs of length n (cf. $T(n, k)$ for Word-RAM.) algorithms.

7.5 Multitape TMs

There are a number of seemingly arbitrary choices in the definition of a TM (albeit less so than in the Word-RAM). We want to argue that these choices don't affect the power of the model. For example, what about multiple tapes?



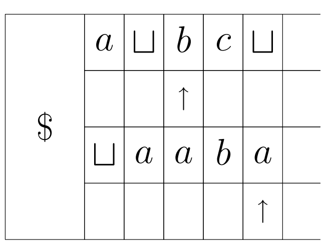
2-tape TM

Convention: First tape used for I/O, like standard TM; Other tapes available for scratch work.

Formally, a k-tape TM has a transition function $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$.

Simulation of multiple tapes by one

- Simulate a k-tape TM by a one-tape TM whose tape is split (conceptually) into $2k$ tracks:
  - k tracks for tape symbols
  - k tracks for head position markers (one in each track)



- To simulate one move of the k-tape TM:
  - Start with the head on the left endmarker
  - Scan down the tape, remembering in the finite control the symbols "scanned" by the k heads
  - Scan back up the tape, revising each track in the vicinity of its head marker
  - Return the head to the left endmarker
- Also need transitions to reformat input and output at start and end of computation.

Speed of the Simulation. Note that the "equivalence" in ability to solve computational problems does not mean comparable speed.

> e.g. A standard TM can recognize palindromes in time $O(n^2)$ (as we saw), and it is known that this is the best possible. But there is an $O(n)$-time 2-tape TM for the same problem:

And this is tight — the simulation above gives at most a quadratic slowdown:

Theorem 7.1 *If M is a multitape TM, then there is a constant c and 1-tape TM M' such that on every input $w \in \Sigma^*$, M' produces the same output as M (or runs forever if M does), and if M halts in time $T(w) \geq |w|$, then M' halts in time at most $c \cdot T(w)^2$.*

7.6 RAMs vs. TMs

Now we show that, even though TMs may seem much simpler and weaker than Word-RAMs, they are actually equivalent in power.

Theorem 7.2 *For every Word-RAM program P, there is a multitape TM M such that for every input $x = (x_1, \ldots, x_n) \in \mathbf{N}^*$, it holds that $M(\langle x \rangle)$ has the "same behavior" as P on input x and size parameter $k = \max\{x_1, ..., x_n, m\}$ where m is the largest constant occurring in P.*

- *$\langle x \rangle$ denotes any reasonable encoding of k, x over the input alphabet Σ of M. For example, we can take $\Sigma = \{0, 1, \|\}$ and $\langle x \rangle = \langle x_1 \rangle \| \cdots \| \langle x_n \rangle$, where $\langle y \rangle$ denotes the binary representation of $y \in \mathbf{N}$.*
- *If P halts with output $y \in \mathbf{N}^*$, then M will halt with output $\langle y \rangle$.*
- *If P does not halt, then M will also run forever.*

- ***Efficiency:*** *If P runs in time $T = T(k,x) \geq |x|$, then M will run in time at most $T^2 \cdot \mathrm{poly}(\log T, \log k)$.*

Simulating a Word RAM Program by a Multitape TM

- Tape 1: $\$M[0]\|M[1]\|\cdots\|M[S-1]$ (sequence of w-bit binary strings).
- Tape 2: S (in binary)
- Tape 3: w (in unary — sequence of w ones)
- Tape 4: $i \in \{0,\ldots,S-1\}$ such that TM head on tape 1 is within $M[i]$ (in binary)
- Tape 5: $j \in \{0,\ldots,w-1\}$ such that TM head on tape 1 is at the j'th bit of $M[i]$ (in unary)
- Tapes 6–(r+5): Registers $R[0],\ldots,R[r-1]$ (in binary)
- Tape 7: scratch tape
- Keep track of program counter ℓ in state of M

The TM can carry out each of the following instructions in time $\text{poly}(w)$ where w is the current word size:

- $P_\ell =$ "$R[i] \leftarrow R[j]$ op $R[k]$" for each of the word operations op we allow.
- If $P_\ell =$ "IF $R[i] = 0$, GOTO ℓ"
- HALT

Each of the following can be implemented in time $O(S \cdot w)$, where S is the current space bound:

- $R[i] \leftarrow M[R[j]]$
- $M[R[i]] \leftarrow R[j]$
- MALLOC

Throughout the computation $S \leq T$ (here we use $T \geq n$) and $w \leq \lceil \log \max\{k+1, S\} \rceil \leq \log \max\{k+1, T\}$. ∎

7.7 The Church–Turing Thesis

Many other models of computation are equivalent in power to Turing Machines:

- TMs with 2-dimensional tapes
- Word RAMs
- Integer RAMs
- General Grammars
- 2-counter machines
- Church's λ-calculus (μ-recursive functions)
- Markov algorithms
- Your favorite programming language (C, Python, OCaml, ...)

The equivalence of each to the others is a mathematical theorem. That these formal models of algorithms capture our intuitive notion of algorithms is the **Church–Turing Thesis**. (Church's thesis = partial recursive functions, Turing's thesis = TMs.) The Church–Turing Thesis is an extramathematical proposition, not subject to formal proof.

The Extended Church-Turing Thesis: Every "reasonable" model of computation can be simulated on a Turing machine with only a polynomial slowdown.

Counterexamples?

- Integer RAMs.
- Randomized computation.
- Parallel computation.
- Analog computers.
- DNA computers.
- Quantum computers.

$\rightarrow$ Extended C-T Thesis needs to be qualified with "sequential and deterministic" and "bounded work per step".

7.8 Languages and Complexity Classes

In classifying computational problems as solvable vs. efficiently solvable vs. unsolvable, it is convenient to restriction attention to *decision problems* — ones where the answer is YES or NO. These correspond to computing functions $f : \Sigma^* \to \{0,1\}$ (where 1=YES), or equivalently to deciding whether the input string is a particular *language* $L \subseteq \Sigma^*$ (namely, the set of strings where the answer is YES).

Motivations for focus on decision problems:

- Don't need to worry about intractability because of $|f(x)|$ being long.
- Many computational problems of interest can be reformulated as an essentially equivalent decision problem.

Some Languages:

- $\textsc{Palindromes} = \{x \in \Sigma^* : x = x^R\}$.
- $\textsc{Primes} = \{\langle N \rangle : N \text{ is a prime number}\}$.
- $\textsc{MinimumSpanningTree} = \{\langle G, w \rangle : G \text{ a weighted graph with a minimum spanning tree of weight at most } w\}$.
- $\textsc{Rank} = \{\langle x_1, \ldots, x_n, \ell \rangle : x_1 \text{ is among the smallest } \ell \text{ items in } x_1, \ldots, x_n\}$.
- $\langle \cdot \rangle$ = any "reasonable" encoding as a string. Which choices of encoding can make the difference between a computational problem being solvable or not? What about solvable in polynomial time?
  - Graph as adjacency matrix vs. list of edges?
  - Numbers in binary vs. base 10?
  - Numbers in binary vs. unary?

Solvable Problems:

- A function $f : \Sigma^* \to \Sigma^*$ is called *recursive* or *computable* if there is an algorithm that computes f.
- A language $L \subseteq \Sigma^*$ is called *recursive* or *decidable* if the characteristic function of L is recursive. An algorithm that computes the characteristic function of L is also said to *decide* (membership in) the language L.
- $\mathrm{R} = \{L : L \text{ is recursive}\}$.

- Does not depend on the choice of computational model (TMs vs. Word RAMs vs. Lambda Calculus)!

Efficiently Solvable Problems:

- $L \in \mathrm{TIME}(T(n))$ if there is an algorithm deciding L that runs in time $O(T(n))$.
- Depends on the model of computation!
- So we should really write $\mathrm{TIME}_{\mathrm{RAM}}(T(n))$, $\mathrm{TIME}_{\mathrm{TM}}(T(n))$, etc. $\mathrm{TIME}_{\mathrm{TM}}(\cdot)$ conventionally refers to *multi-tape* TMs.

Polynomial Time:

- $\mathrm{P} = \bigcup_c \mathrm{TIME}_{\mathrm{TM}}(n^c) = \bigcup_c \mathrm{TIME}_{\mathrm{RAM}}(n^c)$
- Same for all "reasonable" models of computation and "reasonable" encodings of inputs.
- Coarse approximation to "efficiently solvable."

We have $\mathrm{CF} \subseteq \mathrm{P} \subseteq \mathrm{R} \subseteq 2^{\Sigma^*}$, where CF is the class of context-free languages (HW3).

Questions:

- Are there non-recursive languages?
- Are there recursive languages not in P?
- Are there "natural" languages not in R? In $\mathrm{R} \setminus \mathrm{P}$?
- Is FACTORING in P?
- Is $\mathrm{MST} \in \mathrm{TIME}_{\mathrm{RAM}}(O(n))$? Is $\mathrm{MST} \in \mathrm{TIME}_{\mathrm{TM}}(O(n))$?
- Can MULTIPLICATION be done in time $O(n)$ on a Word-RAM? On a multitape TM?

$\vdots$

7.9 Describing Algorithms

Formal Description

- Word-RAM code or TM 6-tuple or TM state diagram
- Only when we ask for it!

Implementation/Pseudocode Description

- Prose description of tape/memory contents, head movements (in case of TMs)
- High-level pseudocode and/or prose description of head movements,
- Omit details of states, transition functions, low-level RAM code (but do convince yourself that a TM/Word-RAM can do what you're describing!)
- Like our proofs that a TMs can simulate Multitape TMs can simulate Word-RAMs.

High-Level Description

- Most of the algorithms descriptions we've seen so far.
- Freely use other algorithms we've seen as subroutines.
- Provide enough detail to be convinced of correctness and runtime on a Word RAM (possibly using some implementation-level specification or pseudocode to make things precise).
- Track number of executions of high-level steps, time for high-level steps (depends on size of data being manipulated).