Codes for Editing Errors

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Based on (1) Haeupler and Shahrasbi – FOCS 2017
(2) Haeupler, Shahrasbi, S. – ICALP 2018

Edit Distance & Codes

- $X \in \Sigma^n$ and $Y \in \Sigma^m$:
  - $X \rightarrow_{\Delta,\Gamma} Y$: If deleting $\leq \Delta$ symbols from X and then inserting $\leq \Gamma$ symbols gives Y.

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  - $E: \Sigma^k \rightarrow \Sigma^n$; $D: \Sigma^* \rightarrow \Sigma^k$

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Unique decoding

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  List decoding

  - $E: \Sigma^k \rightarrow \Sigma^n$; $D: \Sigma^* \rightarrow \binom{\Sigma^k}{L}$

    $\forall X \in \Sigma^k, Y \in \Sigma^*$ $s.t. E(X) \rightarrow_{\Delta,\Gamma} Y, \quad D(Y) \ni X$

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- Given family of code $E = \left(E_n\colon \Sigma^{k_n} \to \Sigma^n\right)_n$
  - Rate = $R = \lim_{n\to\infty} \frac{k_n}{n}$;

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- Questions: what (δ, γ, R) achievable?
  - With list-decoding?
  - While $q = |\Sigma| = O(1)$
  - Algorithmically?

Brief History

- Notion dates back to ‘70s: Levenstein
- [Schulman-Zuckerman ‘90s]:
  $q = poly(n)$, $R \rightarrow 1 - (\delta + \gamma)$ unique decoding
  - Algorithmic!

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- [Haeupler-Shahrasbi-S.'18]
  - $q = O_{\gamma,\delta,R}(1)$; $R \to 1 - \delta$ list decoding

Schulman-Zuckerman Idea

- Indexing:
  - $E: \Sigma^k \to \Sigma^n \quad \to \quad E': \Sigma^k \to (\Sigma \times [n])^n$
    - $E'(x)_i = (E(x)_i, i)$

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    - Insertion (E') $\Rightarrow$ Erasure (E)
    - Deletion (E') $\Rightarrow$ Erasure (E)
    - Same location Ins.+Del. (E') $\Rightarrow$ Error(E)
    - E has distance $\gamma + \delta \Rightarrow E'$ is (δ, γ)-code.

Haeupler-Shahrasbi strategy

- Index with string $S = (S_1, \ldots, S_n) \in [c]^n$, w. $c = O(1) \ll n$
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    - $E'(x)_i = (E(x)_i, S_i)$
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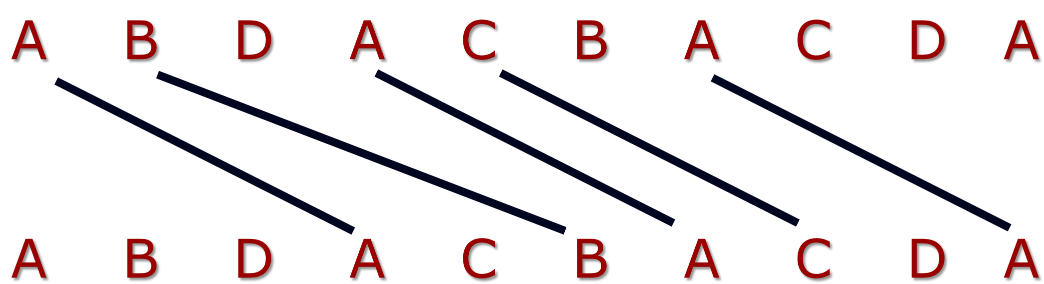
- No longer have $i \neq j \Rightarrow S_i \neq S_j$
- What properties of S useful? Attainable? Verifiable/Constructible?

Synchronization Strings: Defn.

- ϵ-self-matching:
  - $M = \{(i_t, j_t) | \, 1 \le t \le m\} \subseteq [n] \times [n]$ is S-matching if
    - Matching: $i_1, \ldots, i_m$ distinct, $j_1, \ldots, j_m$ distinct
    - Non-trivially S-valid: $S_{i_t} = S_{j_t}$ but $i_t \neq j_t \; \forall t$
    - Monotone: $i_a < i_b \;\Rightarrow j_a < j_b$
  - S ϵ-synch. string if $\forall\, S$-matching M, $|M| \le \epsilon \cdot n$



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- How do they help correct decoding errors?

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  - $C \subseteq \Sigma^n$ is (δ, L)-list-decodable if

    for every $y = (y_1, \dots, y_n) \in \Sigma^n$

    if $S = \{x \in C | \, \#\{i | \, x_i \neq y_i\} \leq \delta . n\}$,

    then $|S| \leq L$

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    then $|S| \leq L$
    (S can be computed efficiently given $Y_1, \ldots, Y_n$)
  - Theorem [Guruswami-Rudra'06]+followups: $\forall \delta, \epsilon > 0, \ell \ \exists \ \Sigma, L$ s.t. $\exists$ a family of (ℓ, δ, L)-list-recoverable codes of rate $1 - \delta - \epsilon$
    - "Folded-Reed-Solomon" codes + Guruswami-Indyk alphabet reduction.

List-decodable codes for editing errors

- Theorem: Let $E: \Sigma^k \to \Sigma^n$ be $(\ell, \delta + \epsilon, L)$-list-recoverable. Let $S \in [c]^n$ be ϵ'-synch string. Then $E': \Sigma^k \to (\Sigma \times [c])^n$ given by $E'(x)_i = (E(x)_i, S_i)$ is (δ, γ)-list-decodable-code.

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  - Let $B = (b_1, \ldots, b_m)$; $Y_1 = \cdots = Y_n = \emptyset$
  - For ℓ iterations do:
    - Let M be largest monotone matching between B and S
    - Removed matched part from B; add matched a-symbols into Y_is.
      - $b_j \leftrightarrow S_i \Rightarrow Y_i \leftarrow Y_i \cup \{a_j\}$

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$\forall \delta, \gamma, \epsilon \;\exists L, \Sigma, \epsilon', \ell, c$ s.t.

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      - $b_j \leftrightarrow S_i \Rightarrow Y_i \leftarrow Y_i \cup \{a_j\}$
  - List-Recover from $Y_1, \ldots, Y_n$

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- Say transmitted $E'(x)$. Received $\left(a_j, b_j\right)_{j \in [m]}$
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  - Case 1: b_j matched to $S_{i'}$ for $i' \neq i$:
    - ϵ'-synch string $\Rightarrow$ $\epsilon' n$ such errors per iteration.
    - ℓ-iterations $\Rightarrow$ $\ell\epsilon' n \leq \frac{\epsilon n}{2}$ such errors.

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  - Case 2: b_j unmatched at end:
    - Let αn code symbols be unmatched at end. Then each iteration matched $\geq \alpha n$ symbols. So $\ell \alpha n \leq m \leq (1+\gamma)n \Rightarrow \alpha n \leq \frac{(1+\gamma)}{\ell} n \leq \frac{\epsilon n}{2}$

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Further work

- Editing + Interaction errors!
- Binary codes for edit distance: Positive rate limits known. [Guruswami-Haeupler-Shahrasbi]
- R $vs.$ δ vs γ for binary codes?

Thank You!