Scattering I

R. Wordsworth

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1 Rayleigh scattering

Here we roughly follow a derivation given in Liou, *An Introduction to Atmospheric Radiation*, which can be referred to for more details. First, we note that Rayleigh scattering involves changes to an oscillating electromagnetic field (the impinging light) due to dipole oscillations induced in a molecule. Basic physics tells us that the dipole induced in a molecule by an electromagnetic field is $\mathbf{p}_0 = \alpha_p \mathbf{E}_{ext}$, where α_p is the polarizability constant. α_p has units of volume. In general, it is a tensor but here we assume it is a scalar for simplicity.

The scattered electromagnetic field far from an oscillating dipole (a result due to Hertz in the late 19th century) is

$$\mathbf{E} = \frac{1}{c^2}\frac{1}{r}\frac{\partial^2 \mathbf{p}}{\partial t^2} sin\theta_0. \tag{1}$$

θ_0 is the angle between the scattered dipole moment and the observation direction, r is distance to the observer, c is speed of light. $\mathbf{p}$, the scattered dipole moment in the electric field direction, can be written as

$$\mathbf{p} = \mathbf{p}_0 e^{-ik(r-ct)} = \alpha_p \mathbf{E}_{ext} e^{-ik(r-ct)}. \tag{2}$$

Hence

$$\frac{\partial^2 \mathbf{p}}{\partial t^2} = -k^2 c^2 \alpha_p \mathbf{E}_{ext} e^{-ik(r-ct)} \tag{3}$$

and

$$\mathbf{E} = -\frac{k^2}{r}\alpha_p \mathbf{E}_{ext} e^{-ik(r-ct)} sin\theta_0. \tag{4}$$

where θ_0 is the angle between the scattered dipole moment and the observer.

We can define *perpendicular* (to solar beam) and *parallel* components

$$E_\perp = -\frac{k^2}{r}\alpha_p E_{\perp,ext} e^{-ik(r-ct)} sin\theta_1 \tag{5}$$

$$E_\parallel = -\frac{k^2}{r}\alpha_p E_{\parallel,ext} e^{-ik(r-ct)} sin\theta_2. \tag{6}$$

$\theta_1 = \pi/2$ always because the scattered dipole moment is normal to the scattering plane. Also $\theta_2 = \pi/2 - \Theta$. This means that

$$E_\perp = -\frac{k^2}{r}\alpha_p E_{\perp,ext} e^{-ik(r-ct)} \tag{7}$$

$$E_{\parallel} = -\frac{k^2}{r}\alpha_p E_{\parallel,ext} e^{-ik(r-ct)} cos\Theta \tag{8}$$

Now note that intensity is proportional to the square of the electric field strength

$$\mathcal{I} = \frac{c|E|^2}{8\pi} \tag{9}$$

hence

$$\mathcal{I}_{\perp} = \frac{k^4}{r^2}\alpha_p^2 \mathcal{I}_{\perp,ext} \tag{10}$$

$$\mathcal{I}_{\parallel} = \frac{k^4}{r^2}\alpha_p^2 \mathcal{I}_{\parallel,ext} cos^2\Theta \tag{11}$$

Now note for unpolarized light (sunlight) $\mathcal{I}_{\perp,ext} = \mathcal{I}_{\parallel,ext} = \mathcal{I}_{ext}/2$. Hence adding we get

$$\mathcal{I} = \frac{\mathcal{I}_{ext}}{r^2}\alpha_p^2 k^4 \frac{1 + cos^2\Theta}{2} \tag{12}$$

$$\frac{\mathcal{I}}{\mathcal{I}_{ext}} = \frac{8\pi^4}{r^2}\alpha_p^2 \lambda^{-4}(1 + cos^2\Theta). \tag{13}$$

This is the Rayleigh scattering formula. We can also define a *phase function*

$$\int_0^{2\pi}\int_0^{\pi} \mathcal{P}(cos\Theta) sin\Theta d\Theta d\phi = 4\pi \tag{14}$$

that expresses the relative scattering angle of the light. For Rayleigh scattering, we can write

$$\mathcal{P}(cos\Theta) = \frac{3}{4}(1 + cos^2\Theta). \tag{15}$$

Because

$$\int_0^{\pi} (1 + cos^2\Theta) sin\Theta d\Theta = \frac{8}{3}, \tag{16}$$

substituting $\mathcal{P}(cos\Theta)$ into our intensity / spectral irradiance equation gives us

$$\frac{\mathcal{I}}{\mathcal{I}_{ext}} = \frac{128\pi^5}{3r^2}\alpha_p^2 \lambda^{-4} \frac{\mathcal{P}(cos\Theta)}{4\pi}. \tag{17}$$

Finally, we can write the incoming flux as $\mathcal{F}_{ext} \equiv \mathcal{I}_{ext}\Delta\Omega$, where $\Delta\Omega$ is the solid angle of the incident radiation. The total scattered Rayleigh *power* for monochromatic light is

$$P_R = \int_{\Omega} \mathcal{I}\Delta\Omega r^2 d\Omega \tag{18}$$

where the integral is over solid angle.

$$P_R = \frac{128\pi^5\alpha_p^2}{3\lambda^4}\mathcal{I}_{ext}\Delta\Omega \int_{\Omega} \frac{\mathcal{P}(cos\Theta)}{4\pi} d\Omega \tag{19}$$

but

$$\int_{\Omega} \frac{\mathcal{P}(cos\Theta)}{4\pi} d\Omega \equiv 1 \tag{20}$$

so

$$P_R = \frac{128\pi^5\alpha_p^2}{3\lambda^4}\mathcal{F}_{ext}. \tag{21}$$

Finally, the Rayleigh cross-section

$$\sigma_R \equiv \frac{P_R}{\mathcal{F}_{ext}} = \frac{128\pi^5\alpha_p^2}{3\lambda^4}. \tag{22}$$

Now α_p is given by the *Lorentz-Lorenz* formula

$$\alpha_p = \frac{3}{4\pi n_s}\frac{m^2-1}{m^2+2}. \tag{23}$$

See The Feynman Lectures vol. 2 for a good derivation of this.

n_s is the number of scatterers (here, molecules) per unit volume. m is refractive index, a complex number. In practice for visible radiation interacting with gases in planetary atmospheres, $Im[m] = m_i \approx 0$ and $Re[m] = m_r$ is close to unity, so

$$\alpha_p \approx \frac{1}{4\pi n_s}(m_r^2 - 1). \tag{24}$$

Finally we can rewrite

$$\sigma_R = \frac{8\pi^3(m_r^2-1)^2}{3n_s^2\lambda^4}f(\delta). \tag{25}$$

The anisotropy factor $f(\delta)$ has been added to account for the fact that α_p is actually a tensor. For Earth air, $f(\delta) \approx 1.061$.