EPS 237 Lesson 14: Scattering in the radiative transfer equation

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1 Aerosol mass extinction and optical depth

The extinction cross section is

$$\sigma_e = \pi a^2 Q_e \tag{1}$$

with a radius and Q_e the extinction efficiency. This relates to the mass extinction coefficient κ_e as

$$\kappa_e = \sigma_e \times \frac{particles}{kg} = \frac{\sigma_e}{M_p}. \tag{2}$$

where M_p is the mass of one particle. Hence

$$\kappa_e = \frac{\pi a^2 Q_e}{\frac{4}{3}\pi a^3 \rho_p} \tag{3}$$

$$\kappa_e = \frac{3Q_e}{4a\rho_p}. \tag{4}$$

An identical expression can be derived for the scattering coefficient κ_s. This is a *really important formula*. It tells us that for a given mass path of particles u, the total optical depth $\tau \propto \kappa_e u$ increases as the particle radius decreases, as long as Q_e remains constant. Why? Because cross-section area decreases less slowly than particle volume as a is decreased.

2 Three key parameters for multiple scattering

The three key scattering parameters for climate calculations are optical depth, single scattering albedo, and the asymmetry parameter:

$$\tau_s = \kappa_s u_s \tag{5}$$

$$\tilde{\omega} = \frac{Q_s}{Q_e} \tag{6}$$

$$\tilde{g} = \frac{1}{4\pi}\int \mathcal{P}(\Theta) cos\Theta d\Omega \tag{7}$$

$\tilde{\omega}$ varies between 0 and 1, while $\tilde{g}$ varies between -1 and 1 (but is rarely less than 0 in practice).

Recall from Lessons 12 and 13 that the phase function $\mathcal{P}(\Theta)$ expresses the fraction of radiation that gets scattering in a given direction, and Θ is the angle between the initial and scattered

radiation directions. Hence if all the radiation keeps going forward when scattered, $\tilde{g}$ will be 1. If all the radiation were reflected backward, as would be the case if particles were tiny mirrors, $\tilde{g}$ would be -1. And in the case of symmetric scattering (as for Rayleigh scattering, for example), $\tilde{g} = 0$.

For monodisperse (single radius) spherical particle populations, these three properties can be calculated directly from Mie theory. For more general particle populations, averaging over the particle size distribution is generally required. See e.g. Petty, *A First Course in Atmospheric Radiation* for more details.

3 The radiative transfer equation with scattering

We recall that in the longwave, if we can assume local thermodynamic equilibrium (LTE), the only source of diffuse photons is $\mathcal{J}_{ir} = B_\nu[T(\tau)]$, the Planck source function. Then we have (in the azimuth-independent case)

$$\mu \frac{d\mathcal{I}(\tau,\mu)}{d\tau} = -\mathcal{I}(\tau,\mu) + B_\nu(\tau). \tag{8}$$

In the shortwave, things are more complicated.

The first thing to take into account is photons scattered from direct stellar radiation. Think about the propagating stellar beam. How much gets to a given layer? If incoming flux is $\mathcal{F}_\odot$, result is $\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0}$, given stellar zenith angle θ_0. Now how much stellar light is scattered? Well, the total extinction cross-section $\kappa_e = \kappa_a + \kappa_s$. From the definition of the single scattering albedo and equation (4),

$$\tilde{\omega} = \frac{\kappa_s}{\kappa_e}. \tag{9}$$

Finally, the phase function $\mathcal{P}(\mu, \mu_0)$ tells us how EM radiation is redistributed as a function of angle. Putting things together gives us a source term for diffuse photons[1]

$$\mathcal{J}_{stel} = \tilde{\omega} \frac{\mathcal{P}(\mu,\mu_0)}{4\pi} \mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0}. \tag{10}$$

Inclusion of $\tilde{\omega}$ makes sure we are only including photons that get scattered (not absorbed). Remember τ is now the *extinction* optical depth. The 4π is a normalization factor for $\mathcal{P}$. Note the units: $\mathcal{P}$ and $\tilde{\omega}$ are dimensionless, while $\mathcal{F}_\odot/4\pi$ has the same units as $\mathcal{J}_{stel}$.

We're not finished: we also need to account for the diffuse (multiple scattering) source. This can be got by integrating $\mathcal{I}$ times the phase function over all angles:

$$\mathcal{J}_{diff} = \frac{\tilde{\omega}}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathcal{I}(\tau,\mu',\phi') \mathcal{P}(\mu,\phi;\mu',\phi') d\mu' d\phi' \tag{11}$$

If we assume the radiation does not depend on azimuth ϕ, we can simplify this to

$$\mathcal{J}_{diff} = \frac{\tilde{\omega}}{2} \int_{-1}^{1} \mathcal{I}(\tau,\mu') \mathcal{P}(\mu,\mu') d\mu'. \tag{12}$$

[1]Note that this expression can also be derived from the integral form of $\mathcal{J}_{diff}$, by writing $\mathcal{I}_\odot = \delta(\mu'-\mu_0)\delta(\phi'-\phi_0)\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0}$.

Hence our full RTE including scattering then becomes

$$\mu \frac{d\mathcal{I}(\tau,\mu)}{d\tau} = -\mathcal{I}(\tau,\mu) + \mathcal{J}_{stel} + \mathcal{J}_{diff} \quad (13)$$

$$\mu \frac{d\mathcal{I}(\tau,\mu)}{d\tau} = -\mathcal{I}(\tau,\mu) + \frac{\tilde{\omega}}{4\pi}\mathcal{P}(\mu,\mu_0)\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} + \frac{\tilde{\omega}}{2}\int_{-1}^{1}\mathcal{I}(\tau,\mu')\mathcal{P}(\mu,\mu')d\mu'. \quad (14)$$

This is progress, but we are still left with a complex integrodifferential equation. To get analytical results or even do a numerical simulation, we need to make further simplifications. One way to make progress is a Monte Carlo approach. Another is to discretize in μ and replace the integral with a sum. The lowest resolution limit of this later approach is the two-stream equations with scattering.

4 The two-stream equation with scattering

We can write, as before,

$$\mathcal{F}_+ = \int_{\Omega_+} \mathcal{I} cos\theta d\Omega \quad (15)$$

$$\mathcal{F}_- = \int_{\Omega_-} \mathcal{I} cos\theta d\Omega \quad (16)$$

This leads in general (in the shortwave only) to

$$\frac{d\mathcal{F}_+}{d\tau} = -\gamma_1\mathcal{F}_+ + \gamma_2\mathcal{F}_- + \gamma_3\tilde{\omega}\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} \quad (17)$$

$$\frac{d\mathcal{F}_-}{d\tau} = -\gamma_2\mathcal{F}_+ + \gamma_1\mathcal{F}_- - \gamma_4\tilde{\omega}\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} \quad (18)$$

where upward and downward streams are now coupled, as they must be. $\gamma_3+\gamma_4 = 1$ by conservation of energy. Other values depend on the parameterization used (see the table in slides and Meador and Weaver, 1980).

5 Planetary albedo

Two-stream equations with scattering but no Planck source

$$\frac{d\mathcal{F}_+}{d\tau} = -\gamma_1\mathcal{F}_+ + \gamma_2\mathcal{F}_- + \gamma_3\tilde{\omega}\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} \quad (19)$$

$$\frac{d\mathcal{F}_-}{d\tau} = -\gamma_2\mathcal{F}_+ + \gamma_1\mathcal{F}_- - (1-\gamma_3)\tilde{\omega}\mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0}. \quad (20)$$

(**Note:** We are working with the Principles of Planetary Climate convention for τ here which means that signs are changed in this equation vs. e.g. Meador & Weaver 1980.) If scattering is conservative, $\tilde{\omega} = 1$, so doing difference of these equations we get

$$\frac{d(\mathcal{F}_+ - \mathcal{F}_-)}{d\tau} = -(\gamma_1-\gamma_2)(\mathcal{F}_+ + \mathcal{F}_-) + \mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0}. \quad (21)$$

But $\gamma_1 = \gamma_2$ for conservative scattering, so we can integrate to get

$$\mathcal{F}_+ - \mathcal{F}_- - \mu_0 \mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} = const. \tag{22}$$

At the upper boundary, $\mathcal{F}_- = 0$, $\mathcal{F}_+ = \mathcal{F}_{+,\infty}$, so

$$const. = \mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot \tag{23}$$

Hence

$$\mathcal{F}_+ - \mathcal{F}_- = \mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot [1 - e^{(\tau-\tau_\infty)/\mu_0}]. \tag{24}$$

Now do the sum. Assume symmetric (e.g. Rayleigh) scattering, so $\tilde{g} = 0$ and $\gamma_3 = \frac{1}{2}$ (see Meador and Weaver table). Now

$$\frac{d(\mathcal{F}_+ + \mathcal{F}_-)}{d\tau} = -2\gamma_1(\mathcal{F}_+ - \mathcal{F}_-) \tag{25}$$

Sub in previous result and drop the 1 on γ_1 to get

$$\frac{d(\mathcal{F}_+ + \mathcal{F}_-)}{d\tau} = -2\gamma(\mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot [1 - e^{(\tau-\tau_\infty)/\mu_0}]) \tag{26}$$

So

$$\mathcal{F}_+ + \mathcal{F}_- = -2\gamma(\mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot)\tau - 2\gamma\mu_0^2 \mathcal{F}_\odot e^{(\tau-\tau_\infty)/\mu_0} + const. \tag{27}$$

but at the top of the atmosphere $\tau = \tau_\infty$, $\mathcal{F}_+ = \mathcal{F}_{+,\infty}$ and $\mathcal{F}_- = 0$ so

$$\mathcal{F}_{+,\infty} + 2\gamma(\mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot)\tau_\infty + 2\gamma\mu_0^2 \mathcal{F}_\odot = const. \tag{28}$$

Hence

$$\mathcal{F}_+ + \mathcal{F}_- = \mathcal{F}_{+,\infty} + 2\gamma(\mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot)(\tau_\infty - \tau) + 2\gamma\mu_0^2 \mathcal{F}_\odot \mathcal{S} \tag{29}$$

where $\mathcal{S} \equiv 1 - e^{(\tau-\tau_\infty)/\mu_0}$ is the part of the direct beam that gets scattered.

At the ground, if surface albedo is zero, $\mathcal{F}_{+,0} = 0$ and

$$\mathcal{F}_{-,0} = \mathcal{F}_{+,\infty} + 2\gamma(\mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot)\tau_\infty + 2\gamma\mu_0^2 \mathcal{F}_\odot \mathcal{S}_0 \tag{30}$$

From the difference equation

$$-\mathcal{F}_{-,0} = \mathcal{F}_{+,\infty} - \mu_0 \mathcal{F}_\odot \mathcal{S}_0. \tag{31}$$

Adding these, we get

$$0 = 2\mathcal{F}_{+,\infty} + 2\gamma\tau_\infty \mathcal{F}_{+,\infty} - 2\gamma\tau_\infty \mu_0 \mathcal{F}_\odot + 2\gamma\mu_0^2 \mathcal{F}_\odot \mathcal{S}_0 - \mu_0 \mathcal{F}_\odot \mathcal{S}_0 \tag{32}$$

or

$$A_a = \frac{\mathcal{F}_{+,\infty}}{\mu_0 \mathcal{F}_\odot} = \frac{\gamma\tau_\infty + \mathcal{S}_0\left(\frac{1}{2} - \gamma\mu_0\right)}{1 + \gamma\tau_\infty}. \tag{33}$$

[not uploaded]

What happens as $\tau_\infty \to 0$ (optically thin limit)?

$$\mathcal{S}_0 \approx 1 - 1 + \tau_\infty/\mu_0 = \tau_\infty/\mu_0 \tag{34}$$

so

$$A_a \approx (\gamma\tau_\infty + \tau_\infty/\mu_0 \left(\frac{1}{2} - \gamma\mu_0\right))(1 - \gamma\tau_\infty) \tag{35}$$

$$A_a \approx (\gamma\tau_\infty + \tau_\infty/2\mu_0 - \tau_\infty\gamma)(1 - \gamma\tau_\infty) \tag{36}$$

$$A_a \approx \tau_\infty/2\mu_0 \tag{37}$$

Can show downward radiation decays only as $1/\tau_\infty$, because of all the multiple scattering. Think thick white clouds in summer.

$$f = \tilde{g} \tag{38}$$

$$\mathcal{P}(\mu) = \frac{1 - g^2}{(1 + g^2 - 2g\mu)^{3/2}} \tag{39}$$