EPS 237 Lesson 2: Radiation Fundamentals

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1 Blackbody Radiation

We start with a classical physics question: what is the spectral emission of electromagnetic (EM) radiation by a body of a given temperature? To address this, we need to consider how energy is distributed among an ensemble of standing EM waves in an idealized black box (see slides). For a 1D standing wave in a box, the wavenumber is

$$k_x = \pi p_x / L \tag{1}$$

where

$$p_x = 1, 2, 3, ... \tag{2}$$

is an integer. In 3D, $p^2 = |\mathbf{p}|^2 = p_x^2 + p_y^2 + p_z^2$. So the increment in the number of possible standing waves per unit p is

$$dN(p) = \left(\frac{1}{8}\right) 4\pi p^2 dp. \tag{3}$$

The 1/8 factor is because we only take the +ve quadrant in wavenumber space. But $k = |\mathbf{k}| = \pi p / L$, so $p^2 dp = \frac{L^3}{\pi^3} k^2 dk$, with L the length of the box. Hence

$$dN(k) = \frac{V}{2\pi^2} k^2 dk \tag{4}$$

where $V = L^3$ is volume. Also $k = 2\pi/\lambda = 2\pi\nu/c$ so $k^2 dk = 8\pi^3 \nu^2 d\nu / c^3$ and hence

$$dN(\nu) = \frac{4\pi V}{c^3} \nu^2 d\nu. \tag{5}$$

Multiply this by 2 to account for polarization degeneracy. Divide by V and define $\mathcal{N} \equiv N/V$ to get result per unit volume:

$$d\mathcal{N}(\nu) = \frac{8\pi}{c^3} \nu^2 d\nu \tag{6}$$

This is the number of standing waves (modes) or *states* per unit volume at frequency ν. We can define the *density of states*

$$g(\nu) \equiv \frac{d\mathcal{N}(\nu)}{d\nu} = \frac{8\pi}{c^3} \nu^2 \tag{7}$$

The spectral energy distribution of radiation is simply the density of states times the average energy of each state, $\epsilon(\nu) = g(\nu)\overline{E}$.

To make further progress, we now invoke the **Classical Equipartition Theorem**. This theorem tells us that each state gets the same average amount of energy $\overline{E} = k_B T$, with T temperature and k_B Boltzmann's constant. Critically, $\overline{E}$ *is not a fn. of* ν in the classical regime.

Given this assumption,

$$\epsilon(\nu) = \frac{8\pi\nu^2}{c^3} k_B T. \tag{8}$$

This expression yields the Rayleigh-Jeans law. More and more energy can be stored at higher frequencies → ultraviolet catastrophe, as described in the slides. It's hard to argue with the above analysis — it's just geometry in a spectral coordinate system. Maybe the problem is with the classical equipartition theorem?

We now introduce Planck's Law, in order to fix $\overline{E}$. We discretize energy by calculating $E_n = nh\nu$ for each mode. The Boltzmann distribution gives the probability that a single mode has energy $E_n = nhv$ (equivalently, has n photons)

$$P(n) = \frac{e^{-E_n/k_B T}}{\Sigma_n e^{-E_n/k_B T}} \tag{9}$$

Hence the mean energy is

$$\overline{E}_\nu = \Sigma_n E_n P(n) \tag{10}$$

or

$$\overline{E}_\nu = \frac{\Sigma_n E_n e^{-E_n/k_B T}}{\Sigma_n e^{-E_n/k_B T}}. \tag{11}$$

Now write $x \equiv e^{-h\nu/k_B T}$. Then

$$\overline{E}_\nu = h\nu \frac{\Sigma_{n=0}^{\infty} n x^n}{\Sigma_{n=0}^{\infty} x^n} \tag{12}$$

Using series expansions [1], this can be rewritten as

$$\overline{E}_\nu = \frac{h\nu x}{1-x} = \frac{h\nu}{x^{-1}-1} \tag{13}$$

and hence

$$\overline{E}_\nu = \frac{h\nu}{e^{h\nu/k_B T} - 1}. \tag{14}$$

The linear part wins out for low ν, while the exponential tail wins out for high ν. Note we can do expansions assuming $h\nu << k_B T$ to get the classical limit. Specifically,

$$e^{h\nu/k_B T} - 1 \approx h\nu/k_B T. \tag{15}$$

Hence

$$\overline{E}_\nu \approx k_B T, \tag{16}$$

which brings us back to the Rayleigh-Jeans Law again.

[1] Writing out the first few terms yields

$$\overline{E}_\nu = h\nu \frac{0 + x + 2x^2 + 3x^3 ...}{1 + x + x^2 + ...} = h\nu x \frac{1 + 2x + 3x^2 ...}{1 + x + x^2 + ...}.$$

But $1 + x + x^2 + ... = \frac{1}{(1-x)}$ and $1 + 2x + 3x^2 ... = \frac{1}{(1-x)^2}$, so $\overline{E}_\nu = h\nu x \frac{1-x}{(1-x)^2} = h\nu x \frac{1}{1-x}$.

This is the mean energy for a given mode, averaged over all the photons. Just like before, to get the energy density of radiation per unit frequency $\epsilon(\nu)$ we multiply by the density of states $g(\nu)$. We use the fact that $\epsilon(\nu) = \frac{8\pi\nu^2}{c^3}\overline{E}_\nu$ to get

$$\epsilon(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT} - 1}. \tag{17}$$

This is the energy density of radiation per unit frequency per unit volume.

The spectral *radiance* (sometimes called irradiance) is

$$B_\nu \equiv B(\nu) = c\epsilon(\nu)/4\pi \tag{18}$$

because radiation propagates in all directions at speed c. Hence

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT} - 1} \tag{19}$$

This is the power emitted per unit area of the body, per unit solid angle measured, per unit frequency.

2 Wavelength / wavenumber conversion

To relate the blackbody radiation in frequency units to that in wavelength units we can use the fact that the total energy emitted across a given frequency range (or wavelength range) must be the same. Hence

$$\int_{\lambda_1}^{\lambda_2} B_\lambda d\lambda = \int_{\nu_2}^{\nu_1} B_\nu d\nu = -\int_{\nu_1}^{\nu_2} B_\nu d\nu \tag{20}$$

and therefore

$$B_\lambda d\lambda = -B_\nu d\nu. \tag{21}$$

But $c = \nu\lambda$ so $d\nu/d\lambda = -c/\lambda^2$. Hence

$$B_\lambda = \frac{c}{\lambda^2}B_\nu. \tag{22}$$

so

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT} - 1}. \tag{23}$$

3 Stefan's Law

Do the integral over ν.

$$F = \int_0^\infty \pi B(\nu) d\nu \tag{24}$$

$$F = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/k_BT} - 1} \tag{25}$$

$$F = \frac{2\pi h}{c^2}\left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{y^3}{e^y - 1} dy \tag{26}$$

for $y \equiv h\nu/k_BT$. Result is

$$F = \frac{2\pi h}{c^2}\left(\frac{k_BT}{h}\right)^4 \frac{\pi^4}{15}. \tag{27}$$

Finally write

$$F = \sigma T^4 \tag{28}$$

where

$$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}. \tag{29}$$

This is the Stefan-Boltzmann constant! Easily checked.