EPS 237 Lesson 3: Stellar Physics

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1 Stellar composition

What are stars made of, and why? Well, Big Bang nucleosynthesis gives 25% He and 75% H by mass. This is because at high temperature, neutrons and protons exchange with each other. This process froze out at around 10^{10} K with a ratio of 1/5, then neutron decay brought this down to 1/7.

$$\nu_e + n \rightleftharpoons e^- + p \tag{1}$$

$$\overline{\nu}_e + p \rightleftharpoons e^+ + n \tag{2}$$

Here ν_e is electron neutrino and $\overline{\nu}_e$ electron antineutrino. Next, at around 10^9 K you get deuteron (n-p) nucleus formation outcompeting photodissociation:

$$n + p \rightarrow d + \gamma \tag{3}$$

where γ is a photon. Chain reaction up to He^4, which is really stable. But too many protons, so a ton of H left over.

2 The mean interior pressure and temperature of stars

Stars do not have much interior motion relative to their size. Hence the bulk interior force balance is gravitational vs. pressure gradient:

$$m\mathbf{a} = \mathbf{F}_1 + \mathbf{F}_2 = 0 \rightarrow \rho\mathbf{g} = +\nabla p. \tag{4}$$

with $\mathbf{g}$ gravity, ρ density and p pressure. In 1D this becomes

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}. \tag{5}$$

where r is the radial coordinate. This is the equation of *hydrostatic balance* (to be derived for planetary atmospheres a little more systematically in the next lecture).

Next, we're going to use gravitational potential energy to get a relationship between mass, radius and pressure. For two point masses m_1 and m_2, the gravitational potential energy is

$$E_G = -\frac{Gm_1m_2}{r}. \tag{6}$$

This becomes

$$E_G = -\int_0^M \frac{Gm(r)}{r} dm \tag{7}$$

for a mass distribution that varies only with r.

In general one can write

$$E_G = -f\frac{GM_S^2}{R_S} \tag{8}$$

where M_S and R_S are the stellar mass and radius and f depends on the density distribution $\rho(r)$. For constant ρ, $f = \frac{3}{5}$. For decreasing density with radius, f is larger, but not by a huge amount. Here we'll just take it to be unity for simplicity, so $E_G \approx -\frac{GM^2}{R_S}$.

Now we relate this to p using the hydrostatic equation. We use the relation $dm = \rho 4\pi r^2 dr$ to get

$$E_G = -\int_0^{R_S} \frac{Gm(r)\rho(r)}{r^2} 4\pi r^3 dr \tag{9}$$

Substitute in the hydrostatic balance equation (5) to get

$$E_G = \int_0^{R_S} 4\pi r^3 \frac{dp}{dr} dr. \tag{10}$$

Now integrate by parts to get

$$E_G = p(r)4\pi r^3|_0^{R_S} - 3\int_0^{R_S} 4\pi r^2 p(r) dr \tag{11}$$

Look at first term: zero at $r = 0$ and zero at surface ($p(R_S) = 0$). Hence

$$E_G = -3\int_0^{R_S} 4\pi r^2 p(r) dr. \tag{12}$$

We can make this neater by defining the average pressure:

$$\langle p \rangle = \frac{\int_0^{R_S} 4\pi r^2 p(r) dr}{\int_0^{R_S} 4\pi r^2 dr} = \frac{\int_0^{R_S} 4\pi r^2 p(r) dr}{V}. \tag{13}$$

Hence

$$\langle p \rangle = -\frac{1}{3}\frac{E_G}{V}. \tag{14}$$

Finally, subbing in $\frac{4}{3}\pi R_S^3$ for V and our previous expression with $f = 1$ for E_G we get

$$\langle p \rangle \approx \frac{GM_S^2}{4\pi R_S^4}. \tag{15}$$

Throw in the numbers for the Sun and we get 10^{14} Pa or 10^9 bar.

Now for temperature. It turns out we can treat gas in Sun as ideal (electrons and ions). It's hot but still non-relativistic. Colder, denser gases tend to be less ideal. Hence we can use ideal gas law

$$\langle p \rangle = \langle n \rangle k_B T = \frac{\langle \rho \rangle}{m_g} k_B T. \tag{16}$$

m_g is the mean mass of the plasma particles. A pure hydrogen plasma (just protons and electrons) would have $\frac{1}{2}(m_p + m_e) \sim \frac{1}{2}m_p$, with m_p the proton mass. The actual value for the Sun is $m_g \sim 0.61$ amu because of He and other heavier elements.

Now, we combine the ideal gas law with the pressure equation we just derived to get

$$\frac{GM_S^2}{4\pi R_S^4} = \frac{\langle\rho\rangle}{m_g}k_BT \tag{17}$$

$$k_BT = \frac{GM_Sm_g}{3R_S}. \tag{18}$$

Put in the numbers for the Sun, and you should get $T \sim 5 \times 10^6$ K. The value at the core in sophisticated solar models is only about 3 times this (see slides). Central *pressure*, on the other hand, is about 200 times the mean value.

3 Fusion

Proton-proton chain

$$p + p \rightarrow d + e^+ + \nu_e \tag{19}$$

$$p + d \rightarrow {}^3\text{He} + \gamma \tag{20}$$

$${}^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + p + p \tag{21}$$

d, which consists of one proton and one neutron, is deuteron (the nucleus of a deuterium atom). There's also the carbon-nitrogen-oxygen cycle, which happens in more massive stars, but we won't worry about that here.

The first reaction in the chain is *much* slower. Timescales are 5 billion years, 1 second and 300,000 years, roughly. $L_\odot$ is 4×10^{26} W and $M_\odot$ is 2×10^{30} kg, so power per unit mass is about 0.2 mW/kg. Pretty low! Fusion happens at a *much* colder temperature than expected from classical considerations. This is because of quantum tunneling.

$$L_\odot/M_\odot = 4 \times 10^{26}\ \text{W}/2 \times 10^{30}\ \text{kg} = 0.2\ \text{mW/kg} \tag{22}$$

4 Luminosity vs. Mass

How does luminosity depend on the mass of the star? Well, from the last lesson we learnt that blackbody emission scales as σT_E^4. Therefore luminosity

$$L_S = 4\pi R_S^2 \sigma T_E^4 \tag{23}$$

But the virial theorem (just derived) tells us that

$$T = \frac{GM_Sm_g}{3k_BR_S} \tag{24}$$

and also

$$M = \rho V \rightarrow R_S = \left(\frac{3}{4}\frac{M_S}{\pi\rho}\right)^{1/3} \rightarrow R_S \propto M_S^{1/3}. \tag{25}$$

Hence if interior temperature T is proportional to the surface temperature T_E the proportionality goes like

$$L_S \propto R_S^2 T_E^4 \propto R_S^2 T^4, \tag{26}$$

$$L_S \propto R_S^2 M_S^4 R_S^{-4}, \tag{27}$$

$$L_S \propto M_S^{4-2/3} \propto M_S^{3.33}. \tag{28}$$

Conversely smaller stars (red dwarfs) can be extremely old. A more rigorous version of this derivation would link internal to emission temperature, rather than just assuming they are proportional (as we have done here), but it turns out that the relation holds up pretty well against observations. A very important implication of this result is that more massive stars burn through fuel *much* more rapidly, and hence die young.