EPS 237 Lesson 5: The radiative transfer equation

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1 Irradiance and flux

Let $\mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu)$ be the density of photons at $\mathbf{x}$ propagating in direction $\hat{\mathbf{s}}$ at frequency ν. This is *spectral irradiance* or *intensity*. In time dt, this intensity will transport energy

$$\delta E = \mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu) dA_s d\Omega_s d\nu dt \tag{1}$$

Note two definitions:

- If $\mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu) = \mathcal{I}(\hat{\mathbf{s}}, \nu)$, the field is *homogeneous*
- If $\mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu) = \mathcal{I}(\mathbf{x}, \nu)$, the field is *isotropic*

The *flux* $\mathcal{F}(\mathbf{x}, \hat{\mathbf{d}}, \nu)$ is defined as total power flowing across a small area dA_d perpendicular to $\hat{\mathbf{d}}$ at frequency ν. It is distinct from $\mathcal{I}$; in some applications confusing the two does not lead to problems, but mostly it does.

To convert irradiance to flux, we need the projected area of dA_d in the $\hat{\mathbf{s}}$ direction, which is:

$$dA_s = (\hat{\mathbf{d}} \cdot \hat{\mathbf{s}}) dA_d = (\cos\gamma) dA_d \tag{2}$$

where γ is the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{d}}$.

We also must integrate over all possible $\hat{\mathbf{s}}$ directions (equivalently, over solid angle Ω_s), so

$$\mathcal{F}(\mathbf{x}, \hat{\mathbf{d}}, \nu) dA_d = \int_{\Omega_s} \mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu) dA_s d\Omega_s \tag{3}$$

$$\mathcal{F}(\mathbf{x}, \hat{\mathbf{d}}, \nu) = \int_{\Omega_s} \mathcal{I}(\mathbf{x}, \hat{\mathbf{s}}, \nu) \cos\gamma d\Omega_s. \tag{4}$$

This relation will be used in a moment to derive the so-called two-stream radiative transfer equation.

2 The general radiative transfer equation

We now derive the radiative transfer equation in its most general form. From the moment we work with monochromatic (single frequency) quantities. We use $\mathcal{I}$, $\mathcal{F}$ for monochromatic quantities, I, F for ν-averaged quantities.

Considering a beam of photons in a medium in which extinction (absorption + scattering) exists, we can write

$$d\mathcal{I} = -k\mathcal{I} ds \tag{5}$$

where k is proportional to the loss rate of photons. This can be rewritten as

$$d\mathcal{I} = -\kappa\rho\mathcal{I}ds \tag{6}$$

where κ is the *mass extinction cross-section* in m^2/kg. κ arises from whatever is present in the atmosphere: gases, clouds, dust, or anything else. It can depend strongly on frequency, but also on pressure and temperature. Generally we can write:

$$\kappa(\nu, p, T) = \sum_{i=0}^{n} \kappa_i(\nu, p, T) q_i(p) \tag{7}$$

with q the specific concentration of an atmospheric species [kg/kg] and i a sum over all species.

Optical depth

$$d\mathcal{I} = -\kappa\rho\mathcal{I}ds \tag{8}$$

can be rewritten as

$$\frac{d\mathcal{I}}{d\tau} = -\mathcal{I} \tag{9}$$

where $d\tau \equiv \kappa\rho ds$. The trivial solution to this equation is

$$\mathcal{I} = \mathcal{I}_0 \mathrm{e}^{-\tau} \tag{10}$$

where $\mathcal{I}_0$ is the starting irradiance. τ is an incredibly useful dimensionless quantity called the *optical depth*. Often, we define it as varying in the vertical direction, in which case $d\tau = \kappa\rho dz$. Then, the hydrostatic relation $dp = -\rho g dz$ we derived in the last lesson can be used to write

$$d\tau = -\frac{\kappa}{g} dp \tag{11}$$

Integrating from the surface where $\tau = 0$, $p = p_s$, to the top of the atmosphere where $p = 0$ (see slides) we get

$$\tau(p = 0) = \tau_\infty = \frac{\kappa p_s}{g} \tag{12}$$

Remember p/g is just a mass column u in $\mathrm{kg/m}^2$, so $\kappa p_s/g$ is dimensionless. We will use τ very frequently from here onwards.

A source term

Our radiative transfer equation is already useful, but it is not yet complete. To complete it, we add in an arbitrary source term $\mathcal{J}$ to get

$$\frac{d\mathcal{I}}{d\tau} = -\mathcal{I} + \mathcal{J}. \tag{13}$$

This is the general monochromatic equation of radiative transfer. Each quantity is defined at a single frequency ν only.

The above equation is very general, but often we be dealing with situations where quantities only vary with height. Then, we can make the *plane parallel approximation* where we assume that horizontal variations are unimportant. We take τ to vary in the vertical coordinate and write

$$\mu \frac{d\mathcal{I}(\tau,\mu)}{d\tau} = -\mathcal{I}(\tau,\mu) + \mathcal{J}(\tau,\mu) \tag{14}$$

where μ is $\cos\theta$: the angle made between the direction of propagation and the vertical. Note that $ds = dz/\cos\theta = dz/\mu$, which is why the factor of μ appears on the left hand side of this equation. Also, because horizontal variations are neglected, we don't need to worry about the dependence of $\mathcal{I}$ on the azimuthal angle ϕ, only its dependence on the polar angle θ.

In the infrared, we can often assume local thermal equilibrium (LTE) and neglect scattering. Then $\mathcal{J} = \mathcal{J}_{IR} = B_\nu[T(\tau)]$, the Planck source function we derived in Lesson 2. Then, our radiative transfer equation becomes

$$\mu \frac{d\mathcal{I}(\tau,\mu)}{d\tau} = -\mathcal{I}(\tau,\mu) + B_\nu(\tau). \tag{15}$$

In the visible region, things are complicated by scattering and we have an integrodifferential source term. We'll handle this later.

3 The two-stream equation

The two-stream equation is a very useful simplification that we will use a lot in the next few lectures. To derive it, we make use of the fact that Planck blackbody radiation is isotropic (no dependence of radiation on direction), and assert that this will also approximately hold true for $\mathcal{I}$.

Then, we integrate over solid angles to convert irradiances to fluxes. First, we integrate over the upward hemisphere:

$$\mathcal{F}_+ = \int_{\Omega_+} \mathcal{I} \cos\theta d\Omega \tag{16}$$

Note the $\cos\gamma$ term becomes $\cos\theta$ — we are taking the normal component of $\mathcal{I}$ to the vertical.

$$\mathcal{F}_+ = \int_0^{2\pi} \int_0^{\pi/2} \mathcal{I} \cos\theta \sin\theta d\theta d\phi. \tag{17}$$

Given isotropy, we can take $\mathcal{I}$ out of the integral, yielding

$$\mathcal{F}_+ = 2\pi\mathcal{I} \left(-\frac{1}{2}\cos^2\theta \right) |_0^{\pi/2} \tag{18}$$

and hence

$$\mathcal{F}_+ = +\pi\mathcal{I}. \tag{19}$$

This is also why $F = \sigma T^4 = \pi \int B_\nu d\nu$ for blackbody radiation, as discussed in Lesson 2. Similarly,

$$\mathcal{F}_- = \int_0^{2\pi} \int_{-\pi/2}^{0} \mathcal{I} \cos\theta \sin\theta d\theta d\phi \tag{20}$$

$$\mathcal{F}_- = -\pi\mathcal{I}. \tag{21}$$

Now we integrate the whole radiative transfer equation over solid angle $\int_{\Omega_+} d\Omega$ to get

$$\frac{d}{d\tau}\int_{\Omega_+} \mathcal{I}\cos\theta d\Omega = -2\pi\mathcal{I} + 2\pi B_\nu \tag{22}$$

$$\frac{1}{2}\frac{d\mathcal{F}_+}{d\tau} = -\mathcal{F}_+ + \pi B_\nu \tag{23}$$

Looking back at eqn. at start of section with μ, we can identify $\overline{\mu} = \overline{\cos\theta} = \frac{1}{2}$. Other values are possible depending on how we integrated. This is sometimes know as the *diffuse approximation*. Now we fold this into τ by defining an average path length optical depth $\overline{\tau} \equiv \tau/\overline{\mu}$. $\overline{\tau}$ is bigger than τ because photons traveling at an angle relative to the vertical have more air to get through before they exit the atmosphere.

Finally,

$$\frac{d\mathcal{F}_+}{d\overline{\tau}} = -\mathcal{F}_+ + \pi B_\nu \tag{24}$$

$$\frac{d\mathcal{F}_-}{d\overline{\tau}} = +\mathcal{F}_- - \pi B_\nu \tag{25}$$

These are our infrared two-stream equations. Note the different signs in the $\mathcal{F}_-$ equation come from the definition $\mathcal{F}_- = -\pi\mathcal{I}$.

4 Solutions to the two-stream equations

We've neglected scattering, which makes these equations uncoupled. Linear inhomogeneous 1st order ODEs in fact. General solution: try $\mathcal{F}_+ = A(\overline{\tau})\mathrm{e}^{-\overline{\tau}}$, which gives for upward branch

$$\frac{d\mathcal{F}_+}{d\overline{\tau}} = \mathrm{e}^{-\overline{\tau}}\frac{dA}{d\overline{\tau}} - \mathcal{F}_+ = -\mathcal{F}_+ + \pi B_\nu \tag{26}$$

so

$$\frac{dA}{d\overline{\tau}} = \pi B_\nu \mathrm{e}^{+\overline{\tau}} \tag{27}$$

hence

$$\int_0^{\overline{\tau}} d(\mathcal{F}_+\mathrm{e}^{+\overline{\tau}'}) = \pi\int_0^{\overline{\tau}} B_\nu(\overline{\tau}')\mathrm{e}^{+\overline{\tau}'}d\overline{\tau}' \tag{28}$$

$$\mathcal{F}_+(\overline{\tau}) = \mathcal{F}_+(0)\mathrm{e}^{-\overline{\tau}} + \pi\int_0^{\overline{\tau}} B_\nu(\overline{\tau}')\mathrm{e}^{\overline{\tau}'-\overline{\tau}}d\overline{\tau}' \tag{29}$$

and for $\mathcal{F}_- = C(\overline{\tau})\mathrm{e}^{+\overline{\tau}}$, yielding

$$\mathcal{F}_-(\overline{\tau}) = \mathcal{F}_-(\overline{\tau}_\infty)\mathrm{e}^{\overline{\tau}-\overline{\tau}_\infty} + \pi\int_{\overline{\tau}}^{\overline{\tau}_\infty} B_\nu(\overline{\tau}')\mathrm{e}^{\overline{\tau}-\overline{\tau}'}d\overline{\tau}'. \tag{30}$$

Heating rate

Heating rates come from thinking about how energy gets deposited in a layer. Per unit optical depth, the relevant quantity is

$$\mathcal{Q} = -\frac{d\mathcal{F}}{d\overline{\tau}} = -\frac{d(\mathcal{F}_+ - \mathcal{F}_-)}{d\overline{\tau}} \tag{31}$$

Sign is necessary because of how we've defined $\overline{\tau}$ (increasing with height). We often want heating rate per unit mass [W/kg or W/Hz/kg] though. To get this, remember $dp = -g\rho dz$. Hence

$$\tilde{\mathcal{Q}} \equiv g\frac{d\mathcal{F}}{dp} = -g\frac{d\overline{\tau}}{dp}\mathcal{Q} \tag{32}$$

Because we're using the two-stream optical depth (includes $\overline{\cos\theta}$),

$$\frac{d\overline{\tau}}{dp} = -\frac{\kappa}{g\overline{\cos\theta}} = -\frac{\kappa}{g\overline{\mu}} \tag{33}$$

So

$$\tilde{\mathcal{Q}} \equiv \frac{\kappa}{\overline{\mu}}\mathcal{Q}. \tag{34}$$

Finite thickness isothermal slab

As an example, we solve the two-stream equations for the case of an isothermal slab floating in space. Define $\overline{\tau} = 0$ at the centre, $\overline{\tau} = \pm\frac{1}{2}\overline{\tau}_b$ at the top and bottom. No incident flux, so $\mathcal{F}_+(-\overline{\tau}_b/2) = \mathcal{F}_-(\overline{\tau}_b/2) = 0$. Furthermore, because the layer is isothermal B_ν is constant and so we can do the integral easily:

$$\mathcal{F}_+ = \pi B_\nu \int_{-\frac{1}{2}\overline{\tau}_b}^{\overline{\tau}} \mathrm{e}^{\overline{\tau}' - \overline{\tau}} d\overline{\tau}' \tag{35}$$

$$\mathcal{F}_+ = \pi B_\nu \left[1 - \mathrm{e}^{-(\overline{\tau} + \frac{1}{2}\overline{\tau}_b)}\right]. \tag{36}$$

The downward version is

$$\mathcal{F}_- = +\pi B_\nu \left[1 - \mathrm{e}^{\overline{\tau} - \frac{1}{2}\overline{\tau}_b}\right]. \tag{37}$$

The heating rate is

$$\mathcal{Q} = -\frac{d(\mathcal{F}_+ - \mathcal{F}_-)}{d\overline{\tau}}, \tag{38}$$

$$\mathcal{Q} = -\pi B_\nu \mathrm{e}^{-\frac{1}{2}\overline{\tau}_b}\left[\mathrm{e}^{-\overline{\tau}} + \mathrm{e}^{\overline{\tau}}\right]. \tag{39}$$

In the optically thin limit we have constant cooling. In the thick limit, we have boundary layer cooling. Condensed water droplets absorb very effectively (more on this later). So the top half of the solution in the thick limit is a reasonable model of how low-lying (stratus) clouds cool to the atmosphere.