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| **CS 221: Computational Complexity** | **Prof. Salil Vadhan** |
| **Lecture Notes 18** | |
| **April 5, 2010** | **Scribe: David Wu** |

# 1 Characterizing IP

Recall that in an interactive proof for a language $L$ we have a computationally unbounded prover $P$ and a verifier $V$ with the properties:

- Efficiency: $V$ runs in time poly($|x|$)
- Completeness: $x \in L \to \Pr[V \text{ accepts in } (P, V)(x)] \geq 2/3$
- Soundness: $x \notin L \to \forall P^*,\ \Pr[V \text{ accepts in } (P^*, V)(x)] \leq 1/3$

Last time we showed that $\mathbf{P}^{\#\mathbf{P}} \subseteq \mathbf{IP}$. In fact:

**Theorem 1** $\mathbf{IP} = \mathbf{PSPACE}$

**Proof:** (sketch)
$\subseteq$ : Homework, PS5
$\supseteq$ : The proof is similar to $\mathbf{P}^{\#\mathbf{P}} \subseteq \mathbf{IP}$, using similar arithmetic techniques to transform the problem into one of polynomials over finite fields, except we use TQBF instead of #SAT. In contrast to the summation in #SAT, handling quantifiers in TQBF causes the degree of polynomial to increase exponentially, so a clever "degree reduction" trick is needed to keep it small.

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## 1.1 Nice Properties of #SAT and TQBF Proof Systems

1. The prover for both can be implemented in $\mathbf{P}^L$ - there is no need for anything stronger. This does not appear to be true for all languages with interactive proofs.

2. Perfect completeness - In both systems if $x \in L$, the verifier accepts always. This implies that every language in **IP** has a perfectly complete interactive proof. (Since $\mathbf{IP} \subseteq \mathbf{PSPACE}$, every language $L \in \mathbf{IP}$ reduces to TQBF, so we can obtain a new, perfectly complete interactive proof that $x \in L$ by reducing $x$ to an instance of TQBF and applying the protocol for TQBF.)

3. Public coins - The verifier in either case needs no hidden randomness. This implies that every language in **IP** has a public-coin protocol, including graph nonisomorphism. (Although public coins may come at the cost of efficiency). Note that the prover still cannot see future coins of the verifier.

## 2 Consequences for Program Checking

**Definition 2**
*A* program checker *(a.ka.* instance checker*) for $f : \{0,1\}^* \to \{0,1\}^*$ is a PPT $M$ such that for all inputs $x$:*

1. *Completeness:* $\Pr[M^f(x) \text{ accepts}] \geq 2/3$ *(or $= 1$ for perfect completeness)*
2. *$\forall g$ such that $g(x) \neq f(x)$,* $\Pr[M^g(x) \text{ accepts}] \leq 1/3$

Idea: someone claims that a program $g$ computes the the function $f$. We want to use $g$ to compute $f$ on an input $x$, but we are concerned that $g$ may be incorrect (either due to bugs or to being malware). By running $M^g(x)$ we can be confident that we won't accept an incorrect value $g(x)$.

**Proposition 3**
*If $L$ and $\overline{L}$ have interactive proof systems where the prover can be implemented in $\mathbf{P}^L$ (or equivalently $\mathbf{P}^{\overline{L}}$), then $L$ has a program checker.*

**Proof:**
Given an oracle $L^*$ to be checked, our program checker is $M^{L^*}(x)$ :

- Query $L^*(x)$ and let $y \in \{0,1\}$ be the result.
- If $y = 1$ simulate the IP for $L$ to verify that $x \in L$.
- If $y = 0$ simulate the IP for $\overline{L}$ to verify that $x \notin L$
- Accept/reject accordingly

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As a result, Graph Isomorphism, #SAT, TQBF all have program checkers because of this. Note that the above does not show that all of **IP** = **PSPACE** has program checkers, because we require that the prover be implementable with oracle access to $L$, rather than to a **PSPACE**-complete problem. In fact, it is an open problem whether SAT has a program checker, and the best known interactive proof for **coNP** still requires a #**P** oracle!

## 3 Arthur–Merlin Games

**Definition 4**
*A* public-coin *interactive proof is an interactive proof $(P, V)$ where each message from $V$ consists of uniformly random coins and at the end $V$ accepts by a deterministic poly-time function of $x$ and the transcript of communications between $P$ and $V$.*
*This is also sometimes known as an Arthur-Merlin protocol, where we imagine Merlin, an all-powerful prover, trying to convince Arthur, the limited verifier, of something.*

**Definition 5**
*For a function $k : \mathbb{N} \to \mathbb{N}$...*
$\mathbf{IP}[k(n)] = \{L : L$ *has interactive proofs with* $\leq k(n)$ *messages*$\}$
$\mathbf{IP} = \bigcup_c \mathbf{IP}[n^c]$
$\mathbf{AM}[k(n)] = \{L : L$ *has public-coin interactive proofs with* $\leq k(n)$ *messages and Arthur speaks first*$\}$
$\mathbf{MA}[k(n)] = \{L : L$ *has public-coin interactive proofs with* $\leq k(n)$ *messages and Merlin speaks first*$\}$
$\mathbf{AM} = \mathbf{AM}[2]$
$\mathbf{MA} = \mathbf{MA}[2]$

We present the following facts:

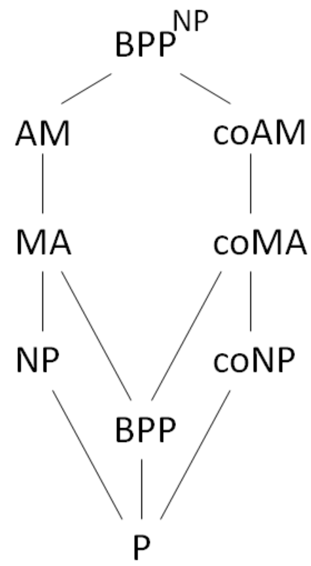
- $\mathbf{IP}[\text{poly}(n)] = \mathbf{AM}[\text{poly}(n)]$ because only public-coins were needed in $\mathbf{IP} = \mathbf{PSPACE}$.
- $\forall k(n) \geq 2$, $\mathbf{IP}[k(n)] = \mathbf{AM}[k(n)]$. In particular, GNI $\in \mathbf{AM}[2]$. Loosely, "public coins = private coins". (We'll prove the case $k(n) = 2$ next time.)
- $\forall k(n) \geq 2$, $\mathbf{MA}[k(n)] \subseteq \mathbf{AM}[k(n)]$. (PS 5)
- $\forall k(n) \geq 2$, $\mathbf{AM}[k(n)] = \mathbf{AM}[k(n)]$ with perfect completeness. (Possibly to be done in section.)
- $\forall k(n) \geq 2$, $\mathbf{MA}[k(n)] = \mathbf{MA}[k(n)]$ with perfect completeness. (Possibly to be done in section.)
- $\forall k(n) \geq 2$, $\forall c$ constant, $\mathbf{AM}[ck(n)] = \mathbf{AM}[k(n)]$. In particular, $\mathbf{AM}[c] = \mathbf{AM}[2]$. (PS 5)

## 3.1 Relationships of AM and MA to NP

In **MA**, we have $M$ sending $m$, then $A$ tossing coins $r$, and then a deterministic verifier $A(x, m, r)$.
By completeness and soundness:
$x \in L \to \Pr_r[\exists m, A(x, r, m) = 1] \geq 2/3$
$x \notin L \to \Pr_r[\exists m, A(x, r, m) = 1] \leq 1/3$
This is exactly **NP** except with a **BPP** verifier, instead of a **P** verifier!

In **AM**, we have $A$ sending coins $r$, then $M$ sending $M$, and then a deterministic verifier $A(x, m, r)$.
By completeness and soundness:
$x \in L \to \Pr_r[\exists m_r, A(x, r, m) = 1] \geq 2/3$
$x \notin L \to \Pr_r[\exists m_r, A(x, r, m) = 1] \leq 1/3$
This is a randomized version of **NP**, where we have some randomness at the beginning, and then afterwards, check an **NP**-like condition that depends on the randomness.

On PS5 you will show $\mathbf{MA} \subseteq \mathbf{AM}$, and thus we have inclusions:




## 4 Approximate Counting $\in$ AM

**Theorem 6**
*For every $f \in \#\mathbf{P}$ and every constant $\alpha > 1$ (or even $\alpha = 1+1/\text{poly}(n)$), we have $GAP_\alpha f \in \mathbf{prAM}$, where $\text{GAP}_\alpha - f$ is the promise problem:*
*yes: $\{(x,t) : f(x) \geq t\}$*
*no: $\{(x,t) : f(x) < t/\alpha\}$*

**Corollary 7**
*Approximate counting and almost-uniform sampling are both in $\mathbf{BPP^{NP}}$.*

**Proof:**
We show the theorem true for $\alpha = 4$. Next time we'll show how to deduce it for $\alpha = 1 + 1/\text{poly}(n)$.

$f \in \#\mathbf{P}$, so by definition $f(x) = |S(x)|$ for some **NP** search problem $S$. We give a **prAM** protocol using hashing. Given $(x,t)$:

1. Arthur chooses $m \in \mathbb{N}$ such that $2^{m-1} > t \geq 2^{m-2}$, picks pairwise-independent hash $h : \{0,1\}^{p(n)} \to \{0,1\}^m$ and sends $h$ to Merlin.

2. Merlin finds $y \in S(x)$ such that $h(y) = 0^m$ and sends $y$.

3. Arthur accepts if $h(y) = 0^m$ and $y \in S(x)$.

Completeness: If $|S(x)| \geq t \geq 2^{m-2}$ then by the Valiant-Vazirani analysis, with probability $\geq 1/8$ there exists some element in $S(x)$ mapping to 0.

Soundness: If $|S(x)| < t/\alpha < 2^{m-1}/\alpha = 2^{m-3}$ then the probability that there exists some element in $S(x)$ mapping to 0 is, by union bound, $\leq \sum_{y \in S(x)} \Pr_h[h(y) = 0^m] = |S(x)|/2^m \leq 2^{m-3}/2^m \leq 1/16$.

Since we have a finite gap 1/8 to 1/16, we can amplify as desired, giving us an **prAM** protocol. ■