LECTURE NOTES FOR INTERLACING FAMILIES

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1. Introduction

In these notes we follow the two papers [6, 7] where Marcus, Spielman and Srivastava developed a method that they coined *the method of interlacing families*. In [6] they proved the existence of infinite families of bipartite Ramanujan graphs for all degrees, and in [7] they provided a positive answer to the notorious Kadison–Singer problem. We take the opportunity to have a slightly more general and unified approach. We consistently use hyperbolic polynomials instead of determinants and barrier function arguments. One benefit of this is that the exposition is self-contained. We don't for example use the Helton–Vinnikov theorem on determinantal representations, instead we use simple concavity properties of hyperbolic polynomials.

2. Graph spectra

In these notes a graph will be (unless explicitly stated) a pair $G = (V, E)$ of finite sets where $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$. Recall that the *adjacency matrix*, $A(G) = (a_{ij}(G))$, of a graph $G = (V, E)$ is the $E \times E$ matrix defined by

$$a_{ij}(G) = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

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The *characteristic polynomial* of G is the characteristic polynomial of $A(G)$, i.e.,

$$\chi_G(x) := \chi_{A(G)}(x) := \det(xI - A(G)),$$

where I denotes the identity matrix. The *spectrum* of G, $\mathrm{Spec}(G)$, is defined to be the multiset of eigenvalues of $A(G)$.

Recall that a graph is *d–regular* if each vertex has degree d. Suppose G is *d–regular*. If $\mathbf{1}_V$ is the vector of all ones, then $A(G)\mathbf{1}_V = d\mathbf{1}_V$, and hence $d \in \mathrm{Spec}(G)$. Let E_{ij} be the matrix with all entries zero but the the (i,j)- and (j,i)-entry which are one. Recall that a symmetric matrix is *positive definite* if all its eigenvalues are positive and *positive semi-definite* if all its eigenvalues are nonnegative. Clearly $E_{ii} + E_{jj} - E_{ij}$ is positive semi-definite (it has eigenvalues 1 and 0). If $t > d$, then

$$tI - A(G) = (t-d)I + dI - A(G) = (t-d)I + \sum_{ij \in E} E_{ii} + E_{jj} - E_{ij},$$

is positive definite since it is the sum of a positive definite matrix and a positive semi-definite matrix. Hence $\det(tI - A(G)) > 0$, so that $t \notin \mathrm{Spec}(G)$. A similar reasoning proves that if $t < -d$, then $t \notin \mathrm{Spec}(G)$.

Example 2.1. The adjacency matrix of the complete graph K_{d+1} is equal to $J - I$, where J is the all ones matrix. From this it is easy to see that $\mathrm{Spec}(K_{d+1}) = \{d, -1, \ldots, -1\}$.

Example 2.2. Recall that a graph is *bipartite* if the vertex set may be partitioned as $V = X \cup Y$ so that each edge is of the form $\{x, y\}$ where $x \in X$ and $y \in Y$. Hence the adjacency matrix of a bipartite graph has a block structure

$$A(G) = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix},$$

where K is a matrix of size $|X| \times |Y|$ and K^T is the transpose of K. Note that $A(G)\mathbf{w} = \lambda\mathbf{w}$, where $\mathbf{w} = (\mathbf{x}^T, \mathbf{y}^T)^T$ and $\mathbf{x} \in \mathbb{R}^{|X|}$ and $\mathbf{y} \in \mathbb{R}^{|Y|}$ if and only if $K\mathbf{y} = \lambda\mathbf{x}$ and $K^T\mathbf{x} = \lambda\mathbf{y}$. Hence

$$A(G)\mathbf{w} = \lambda\mathbf{w} \quad \text{if and only if} \quad A(G)\widetilde{\mathbf{w}} = -\lambda\widetilde{\mathbf{w}}, \text{ where } \widetilde{\mathbf{w}} = (-\mathbf{x}^T, \mathbf{y}^T)^T.$$

Hence $\mathrm{Spec}(G)$ is symmetric around the origin.

3. Expander graphs, Ramanujan graphs and Two-lifts

The *nontrivial eigenvalues* of a d-regular graph are those in the interval $(-d, d)$. A d-regular graph is said to be *Ramanujan* (or d–Ramanujan) if it is connected and all nontrivial eigenvalues lie in the closed interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. Ramanujan graphs are extremal in the following sense.

Theorem 3.1 (Alon–Boppana)**.** *Let d be a positive integer. There exists a constant c such that for each d-regular graph G the largest nontrivial eigenvalue of G is larger than*

$$2\sqrt{d-1} \cdot (1 - c/\Delta^2),$$

where Δ is the diameter of G.

Note that the diameter grows to infinity with the size of the ground set.

The *spectral gap* of a graph is the difference between its largest and second largest eigenvalue. Ramanujan graphs are good *expander graphs* in the following sense. The *edge expansion ratio* of a graph $G = (V, E)$ is defined as

$$h(G) = \min \left\{ \frac{|\partial S|}{|S|} : 0 < |S| \leq \frac{|V|}{2} \right\}, \text{ where}$$
$$\partial S = \{\{u, v\} \in E : u \in S, v \notin S\}.$$

Theorem 3.2 (Dodziuk, Alon-Milman, Alon)**.** *Let $G = (V, E)$ be connected, d regular with spectral gap δ. Then*

$$\delta/2 \leq h(G) \leq \sqrt{2d\delta}.$$

Bilu and Linial suggested a procedure of constructing infinite families of d-Ramanujan graphs by iteratively applying so called 2*-lifts*. If $G = (V, E)$ is a graph and $\mathbf{s} : E \to \{-1, 1\}$ is an assignment of signs to the edges, we define a new graph $G_\mathbf{s} = (V_\mathbf{s}, E_\mathbf{s})$ as follows. The set of vertices $V_\mathbf{s} = V^1 \cup V^2$ is a disjoint union of two copies $V^1 = \{v^1 : v \in V\}$ and $V^2 = \{v^2 : v \in V\}$ of V. The edges of $G_\mathbf{s}$ come in pairs, one pair for each edge $e = \{u, v\} \in E$: If $\mathbf{s}(e) = 1$, then the corresponding edges in $E_\mathbf{s}$ are $\{u_1, v_1\}$ and $\{u_2, v_2\}$. If $\mathbf{s}(e) = -1$, then the corresponding edges in $E_\mathbf{s}$ are $\{u_1, v_2\}$ and $\{u_2, v_1\}$.

Lemma 3.3. *Let $G = (V, E)$ be a graph and $\mathbf{s} : E \to \{-1, 1\}$. Then*

$$\chi_{G_\mathbf{s}}(x) = \chi_G(x) \det(xI - A_\mathbf{s}(G)),$$

where $A_\mathbf{s}(G)$ is the signed adjacency matrix *with entry (i, j) equal to zero if $\{i, j\}$ is not an edge and $\mathbf{s}(i, j)$ if $\{i, j\}$ is an edge.*

Proof. We may order the vertices so that the adjacency matrix of $G_\mathbf{s}$ has the form

$$A(G_\mathbf{s}) = \begin{pmatrix} A_+ & A_- \\ A_- & A_+ \end{pmatrix},$$

where A_+ corresponds to the positive edges, and A_- corresponds to the negative edges. Now write $\mathbf{w} = (\mathbf{x}^T, \mathbf{y}^T)^T$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{|V|}$. Then $A(G_s)\mathbf{w} = \lambda \mathbf{w}$ if and only if

$$\begin{aligned} A_+\mathbf{x} + A_-\mathbf{y} &= \lambda \mathbf{x}, \qquad \text{and} \\ A_-\mathbf{x} + A_+\mathbf{y} &= \lambda \mathbf{y} \end{aligned}$$

Clearly this happens if and only if $A(G)\mathbf{u} = \lambda \mathbf{u}$ and $A_\mathbf{s}(G)\mathbf{v} = \lambda \mathbf{v}$, where $\mathbf{u} = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{y}$. Hence, for each eigenvalue of $A(G)$ we get an eigenvalue of $A(G_\mathbf{s})$ by choosing $\mathbf{v} = 0$. Also the dimension of the corresponding eigenspaces coincide. Similarly, for each eigenvalue of $A_\mathbf{s}(G)$ we get an eigenvalue of $A(G_\mathbf{s})$ by choosing $\mathbf{u} = 0$. Altogether the sum of the multiplicities of these eigenvalues add up to $2|V|$, which proves that all eigenvalues are accounted for. □

Note that if G is d-regular, then so is $G_\mathbf{s}$. Moreover, the new eigenvalues of $G_\mathbf{s}$ are the eigenvalues of the signed adjacency matrix $A_\mathbf{s}(G)$. Hence if G is a Ramanujan graph and the eigenvalues of $A_\mathbf{s}(G)$ are bounded in modulus by $2\sqrt{d-1}$, then $G_\mathbf{s}$ is also Ramanujan. Bilu and Linial conjectured that this is always possible, and hence that one can recursively build infinite families of Ramanjuan graphs:

Conjecture 3.4 (Bilu & Linial)**.** *Suppose G is d-regular and Ramanujan. Then there exists an $\mathbf{s} : E \to \{-1, 1\}$ so that the eigenvalues of $A_{\mathbf{s}}(G)$ lie in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.*

Marcus, Spielman and Srivastava were able to prove this for bipartite graphs, [6].

4. The matching polynomial

A *matching* in a graph $G = (V, E)$ is a subset $M \subseteq E$ such that no two edges in M have a common vertex. The matching polynomial is the generating polynomial for matchings:

$$\mu_G(x) = \sum_{k \geq 0} (-1)^k m_k(G) x^{|V|-2k},$$

where $m_k(G)$ denotes the number matchings of size k in G.

Lemma 4.1. *If $G = (V, E)$ is a graph and $i \in V$, then*

$$\mu_G(x) = x\mu_{G\setminus i}(x) - \sum_{\{i,j\} \in E} \mu_{G\setminus i \setminus j}(x),$$

where $G \setminus i$ is the graph obtained by deleting vertex i and all edges incident to i.

Proof. The number of matchings of size k that do not contain i is equal to $m_k(G\setminus i)$, and the number of matchings that contain i is equal to $\sum_{\{i,j\}\in E} m_{k-1}(G \setminus i \setminus j)$. Hence

$$m_k(G) = m_k(G \setminus i) + \sum_{\{i,j\} \in E} m_{k-1}(G \setminus i \setminus j),$$

from which the lemma follows. □

Theorem 4.2. *Let $G = (V, E)$ be a graph and $\{\mathbf{s}(e)\}_{e \in E}$ independent random variables with values in $\{-1, 1\}$ such that $\mathbb{E}\mathbf{s}(e) = 0$ for all $e \in E$. Then*

$$\mathbb{E} \det(xI - A_{\mathbf{s}}(G)) = \mu_G(x).$$

Proof. Let $G = (V, E)$, where $V = \{1, \ldots, n\}$ be a graph and let G' be the graph obtained by adding loops to each vertex in G. A cycle $v_1, v_2, \ldots, v_k = v_1$ is *simple* if $v_i \neq v_j$ for all $1 \leq i < j \leq k - 1$. A *disjoint cycle cover* of G' is a vertex-disjoint collection $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ of simple cycles (loops and two-cycles are allowed) in G' such that their union is V. Define the *weight* of a cycle $C = v_1, v_2, \ldots, v_k$ to be

$$w(C) = \begin{cases} x & \text{if } C \text{ is a loop,} \\ (-1)^k \mathbf{s}(v_1, v_2)\mathbf{s}(v_2, v_3) \cdots \mathbf{s}(v_{k-1}, v_k) & \text{otherwise.} \end{cases}$$

The weight of a disjoint cycle cover $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ is $w(\mathcal{C}) = w(C_1) \cdots w(C_\ell)$. If we extend $\mathbf{s}$ so that $\mathbf{s}(i, j) = 0$ whenever $\{i, j\} \notin E$, then

$$\begin{aligned} \det(xI + A_{\mathbf{s}}(G)) &= \sum_{\pi \in \mathfrak{S}_n} \operatorname{sign}(\pi) x^{\#\text{fixed points}} \prod_{i \text{ not fixed}} \mathbf{s}(i, \pi(i)) \\ &= \sum_{\mathcal{C}} w(\mathcal{C}), \end{aligned}$$

where the sum is over all disjoint cycle covers of G'. Note that if C is a cycle of length greater than 2, then $\mathbb{E}w(C) = 0$ because $\{\mathbf{s}(e)\}$ are independent and each $\mathbf{s}(e)$ has expectation 0. If C is a cycle of length 2, then $w(C) = -1$. Hence

$$\mathbb{E}\sum_{\mathcal{C}} w(\mathcal{C}) = \sum_{\mathcal{C}=\{C_1,\ldots,C_\ell\}} \mathbb{E}w(C_1)\cdots\mathbb{E}w(C_\ell) = \sum_{\mathcal{C}'} w(\mathcal{C}'),$$

where the latter sum is over all disjoint cycle covers into loops and two-cycles. Clearly such cycle covers are in bijection with matchings. □

The maximum degree of a vertex in of a graph is denoted $\Delta(G)$.

Lemma 4.3. *Let $\delta \geq \Delta(G)$ be an integer greater than 1. If the degree of a vertex i is smaller than δ, then*

$$\frac{\mu_G(x)}{\mu_{G\setminus i}(x)} > \sqrt{\delta-1},$$

whenever $x > 2\sqrt{\delta-1}$.

Proof. Induction over $|V|$. If the degree of i is 0, then the quotient is equal to x, and the lemma follows. Otherwise, by Lemma 4.1 and induction

$$\begin{aligned}\frac{\mu_G(x)}{\mu_{G\setminus i}(x)} &= x - \sum_{\{i,j\}\in E} \frac{\mu_{G\setminus i\setminus j}(x)}{\mu_{G\setminus i}(x)} \\ &\geq 2\sqrt{\delta-1} - (\delta-1)/\sqrt{\delta-1} = \sqrt{\delta-1}.\end{aligned}$$

□

Theorem 4.4 (Heilmann & Lieb)**.** *Suppose $\Delta(G) > 1$. If x is a real number with $|x| > 2\sqrt{\Delta(G)-1}$, then $\mu_G(x) \neq 0$.*

Proof. Note that $\mu_G(x)$ is even or odd, so that we may assume $x > 2\sqrt{\Delta(G)-1}$. By Lemma 4.3

$$\frac{\mu_G(x)}{\mu_{G\setminus i}(x)} = x - \sum_{\{i,j\}\in E} \frac{\mu_{G\setminus i\setminus j}(x)}{\mu_{G\setminus i}(x)} > 2\sqrt{\Delta(G)-1} - \Delta(G)/\sqrt{\Delta(G)-1} > 0,$$

for all $i \in V$, since the degree of j in $G \setminus i$ is smaller than $\Delta(G)$. The proof now follows by induction over $|V|$. □

5. Interlacing families

Let f and g be two real–rooted polynomials of degree $n-1$ and n, respectively. We say that f is an interleaver of g if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n,$$

where $\alpha_1 \leq \cdots \leq \alpha_{n-1}$ and $\beta_1 \leq \cdots \leq \beta_n$ are the zeros of f and g, respectively.

Example 5.1. If f is a real–rooted polynomial of degree at least two, then f' is an interleaver of f. Indeed, by Rolle's theorem there is a zero of f' between each pair consecutive different zeros of f. A multiple zero of f is also a zero of f' (of multiplicity one less) from which the interlacing property follows.

Example 5.2. If A is a hermitian matrix and A' is a maximal sub–matrix of A obtained by deleting row and column i for some i, then the characteristic polynomial of A' is an interleaver of the characteristic polynomial of A.

By the next theorem we see that the zeros of the matching polynomials are real and located in the interval $[-2\sqrt{\Delta(G)-1}, 2\sqrt{\Delta(G)-1}]$.

Theorem 5.1 (Heilmann–Lieb). *Let $G=(V,E)$ be a graph and $i \in V$. Then $\mu_G(x)$ is real–rooted and $\mu_{G\setminus i}(x)$ is an interleaver of $\mu_G(x)$ for all $i \in V$.*

Proof. Induction on $n = |V|$. Assume the theorem is true for n and that $|V| = n+1$. Hence $\mu_{G\setminus i \setminus j}(x)$ is an interleaver of $\mu_{G\setminus i}(x)$ for all $i, j \in V$. Assume first that all zeros $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ of $\mu_{G\setminus i}(x)$ are distinct and that for each $1 \leq k \leq n$ there is a j such that $\{i,j\} \in E$ such that $\mu_{G\setminus i \setminus j}(\alpha_k) \neq 0$. Then by the interlacing property $(-1)^{k-1}\mu_{G\setminus i \setminus j}(\alpha_k) \geq 0$ for all $\{i,j\} \in E$ and $1 \leq k \leq n$. Hence $(-1)^k \mu_G(\alpha_k) > 0$ by Lemma 4.1 for all $1 \leq k \leq n$. It follows that $\mu_{G\setminus i}(x)$ is an interleaver of $\mu_G(x)$. If $\mu_{G\setminus i}(x)$ has a zero α of multiplicity $m > 1$, then by the interlacing property α is a zero of $\mu_{G\setminus i \setminus j}(x)$ of multiplicity at least $m-1$ for all j. Hence we may factor through $(x-\alpha)^{m-1}$ and reduce it to the case of simple zeros. The same type of reduction applies if there is a k such that $\mu_{G\setminus i \setminus j}(\alpha_k) = 0$ for all j such that $\{i,j\} \in E$. □

Lemma 5.2. *Let f and g be real–rooted polynomials. Then the polynomial*

$$f\left(\frac{d}{dx}\right) g(x) = \sum_{k \geq 0} \frac{f^{(k)}(0)}{k!} g^{(k)}(x)$$

is either identically zero or real–rooted.

Moreover f' is an interleaver of $f + \alpha f'$ for all $\alpha \in \mathbb{R}$.

Proof. Clearly it suffices to prove the second statement. Let g be the greatest common divisor of f and f'. Then f'/g is an interleaver of f/g and they have no common zeros. Hence f/g alternates in sign at consecutive zeros of f'/g. The same is true for $f/g + \alpha f'/g$. Hence f'/g is an interleaver of $f/g + \alpha f'/g$. □

Lemma 5.3. *Suppose $\epsilon \in \mathbb{R}$ and f is a real–rooted polynomial of degree n. Then all zeros of*

$$f_\epsilon(x) := \left(1 + \epsilon \frac{d}{dx}\right)^n f(x)$$

are real and distinct.

Proof. Suppose α is a zero of multiplicity $m \geq 2$ of $f + \epsilon f'$. Then α is a zero of f' (since f' is an interleaver of $f + \epsilon f'$ by Lemma 5.2). But then α is a zero of $f = (f + \epsilon f') - \epsilon f'$. Since the multiplicity of α as a zero of f is one greater than the multiplicity of α as a zero of f', it follows that that the multiplicity of α as a zero of f is precisely $m+1$.

We have proved that if f has zero of multiplicity greater than one, then the quantity "highest multiplicity of the zeros" goes down by one when applying $1 + \epsilon d/dx$. This proves the lemma. □

We will on several occasions use the fact that the zeros of a polynomial (or analytic function) depends continuously on its coefficients:

Lemma 5.4 (Hurwitz' theorem). *Let $\{f_n(z)\}_{n=1}^\infty$ be a sequence of functions that are analytic on a connected open set $\Omega \subseteq \mathbb{C}$. Suppose f_n converges to f, uniformly on each compact subset of Ω. If ζ is a zero of $f(z)$ of multiplicity M, then for each $\epsilon > 0$ there exists a number N such that for each $n \geq N$, f_n has exactly M zeros in $\{z \in \Omega : |z - \zeta| < \epsilon\}$ counted with multiplicities.*

Lemma 5.5. *Let f_0 and f_1 be two real–rooted polynomials of the same degree and with positive leading coefficient. The following are equivalent:*

(S) *For all $p \in [0,1]$, the polynomial*
$$f_p(x) := (1-p)f_0(x) + pf_1(x)$$
is real–rooted.

(I) *The polynomials $f_0(x)$ and $f_1(x)$ have a common interleaver.*

Proof. By using Lemma 5.3 and Hurwitz' theorem we may assume that f_0 and f_1 have simple zeros [How?]. By factoring through any common zeros we may also assume that f_0 and f_1 have no zeros in common.

(I) $\Longrightarrow$ (S): By slightly altering h we may assume that the common interleaver, h, has no zeros in common with f_0f_1. Let $\alpha_1 < \cdots < \alpha_{n-1}$ be the zeros of h. Then $(-1)^{n-k}f_p(\alpha_k) > 0$ for all $1 \leq k \leq n-1$. It follows that f_p has a zero in each of the n open intervals cut out by $\{\alpha_i\}_{i=1}^{n-1}$. Hence f_p is real–rooted and h is an interleaver of f_p.

(S) $\Longrightarrow$ (I): Let $\beta_1 < \cdots < \beta_n$ be the zeros of f_0. We claim that for each $1 \leq i \leq n-1$, there is a point $\gamma_i \in (\beta_i, \beta_{i+1})$ such that $f_0(\gamma_i)f_1(\gamma_i) > 0$. The claim implies that the polynomial $\prod_{i=1}^{n-1}(x-\gamma_i)$ is a common interleaver of f_0 and f_1. Suppose that the claim is false for some i, and that (without loss of generality) f_0 is positive on (β_i, β_{i+1}). As p increases from 0, the zero β_i will move to the right and β_{i+1} will move to the left until they coincide for some $0 < p_0 < 1$. When we increase p from p_0 we will lose the two zero in $[\beta_i, \beta_{i+1}]$, and since the zeros depend continuously on the coefficients we have created a pair of non–real zeros contrary to the assumptions. □

If $f(x)$ is a real–rooted polynomial with zeros $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, let $I_k = [\alpha_k, \alpha_{k+1}] \subset \mathbb{R}$ and $B(f) = I_1 \times I_2 \times \cdots \times I_{n-1} \subset \mathbb{R}^{n-1}$. Note that a family $f_1, \ldots, f_k$ of real–rooted polynomials of the same degree have a common interleaver if and only if $B(f_1) \cap \cdots \cap B(f_k) \neq \emptyset$.

We will only need the (almost trivial) $d = 1$ case of Helly's theorem.

Theorem 5.6 (Helly's theorem)**.** *Let $C_1, \ldots, C_k \subseteq \mathbb{R}^d$ be convex sets such that each collection of $d+1$ of them have non-empty intersection. Then $C_1 \cap C_2 \cap \cdots \cap C_k \neq \emptyset$.*

Theorem 5.7. *Let $f_1(x), \ldots, f_m(x)$ be real–rooted polynomials of the same degree and positive leading coefficients. The following are equivalent.*

(1) *For all $1 \leq i < j \leq m$ and $p \in [0,1]$, the polynomial*
$$(1-p)f_i(x) + pf_j(x)$$
is real–rooted.

(2) *For all $1 \leq i < j \leq m$, $f_i(x)$ and $f_j(x)$ have a common interleaver.*

(3) *$f_1(x), \ldots, f_m(x)$ have a common interleaver.*

(4) *for all $p_1, \ldots, p_m \geq 0$, $\sum_i p_i = 1$, the polynomial*
$$p_1f_1(x) + \cdots + p_mf_m(x)$$
is real–rooted.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 5.5, while (2) $\Rightarrow$ (3) follows from Helly's theorem. (3) $\Rightarrow$ (4) is proved exactly as in the proof of Lemma 5.5 and (4) $\Rightarrow$ (1) is immediate. □

Corollary 5.8. *Suppose $f(x)$ and $g(x)$ are polynomials of degree $n+1$ and n respectively, both with positive leading coefficients. The following are equivalent:*

(1) *$f(x) + \alpha g(x)$ is real–rooted for all $\alpha \in \mathbb{R}$.*
(2) *$g(x)$ is an interleaver of $f(x)$.*

Proof. (2) $\Rightarrow$ (1) follows by a standard sign-analysis as above.

For (1) $\Rightarrow$ (2) consider the family $\mathcal{F}_N = \{f_m(x)\}_{m=-N}^{N}$ where $f_m(x) = f(x) + mg(x)$ and $N > 0$ is an integer. By Theorem 5.7 the polynomials in $\mathcal{F}_N$ have a common interleaver. When $m \to \infty$ one zero of $f_m(x)$ will tend to $-\infty$ and the rest of the zeros will approach the zeros of $g(x)$, by Hurwitz' theorem. Similarly as $m \to -\infty$ one zero will tend to ∞ and the rest of the zeros will approach the zeros of $g(x)$. Let $h_N(x)$ be a common interleaver of $\mathcal{F}_N$. It follows that $\lim_{N\to\infty} h_N(x) = g(x)$. Since, by construction, $h_N(x)$ is an interleaver of $f(x)$, we have by Hurwitz' theorem that $g(x)$ is an interleaver of $f(x)$. □

Lemma 5.9. *Let $f_1, \dots, f_m$ be real–rooted polynomials that have the same degree and positive leading coefficients. If $f_1, \dots, f_m$ have a common interleaver, then for some $1 \le i \le m$ the largest zero of f_i is smaller or equal to the largest zero of the polynomial*

$$f_\emptyset := f_1 + f_2 + \cdots + f_m.$$

Proof. If α is the largest zero of the common interleaver, then $f_i(\alpha) \le 0$ for all i, so that the largest zero, β, of $f_\emptyset$ is located in the interval $[\alpha, \infty)$, as are the largest zeros of f_i for each $1 \le i \le m$. Since $f_\emptyset(\beta) = 0$, there is an index i such that $f_i(\beta) \le 0$. Hence the largest zero of f_i is smaller or equal to β. □

Definition 5.1. Let $S_1, \dots, S_m$ be finite nonempty sets and $\{f_{\mathbf{s}}(x)\}_{\mathbf{s}\in S_1\times\cdots\times S_m}$ a set of polynomials of the same degree and with positive leading coefficients. For $s_1 \in S_1, \dots, s_k \in S_k$, where $1 \le k \le m$, define

$$f_{s_1 s_2 \cdots s_k}(x) := \sum_{s_{k+1}\cdots s_m \in S_{k+1}\times\cdots\times S_m} f_{s_1\cdots s_k s_{k+1}\cdots s_m}(x).$$

Also let

$$f_\emptyset(x) := \sum_{\mathbf{s}\in S_1\times\cdots\times S_m} f_{\mathbf{s}}(x).$$

The family $\{f_{\mathbf{s}}(x)\}_{\mathbf{s}\in S_1\times\cdots\times S_m}$ is an *interlacing family* if for each $0 \le k \le m-1$ and $s_1 \in S_1, \dots, s_k \in S_k$ the polynomial $\{f_{s_1\cdots s_k s_{k+1}}\}_{s_{k+1}\in S_{k+1}}$ have a common interleaver.

Theorem 5.10. *Suppose $\{f_{\mathbf{s}}(x)\}_{\mathbf{s}\in S_1\times\cdots\times S_m}$ is an interlacing family. Then there exists an $\mathbf{s} \in S_1 \times \cdots \times S_m$ such that the largest zero of $f_{\mathbf{s}}(x)$ is no larger than the largest zero of $f_\emptyset(x)$.*

Proof. Induction over m. The case $m = 1$ is Lemma 5.9, so suppose $m > 1$. Clearly the family $\{f_{\mathbf{s}'}(x)\}_{\mathbf{s}'\in S_1\times\cdots\times S_{m-1}}$ is an interlacing family. By induction there is a sequence $\mathbf{s}' = s_1 \cdots s_{m-1} \in S_1 \times \cdots \times S_{m-1}$ so that the largest zero of $f_{\mathbf{s}'}(x)$ is no larger then the largest zero of $f_\emptyset(x)$. By definition the polynomials $\{f_{\mathbf{s}' s_m}(x)\}_{s_m\in S_m}$ have a common interleaver so the theorem follows from Lemma 5.9 since

$$f_{\mathbf{s}'}(x) = \sum_{s_m\in S_m} f_{\mathbf{s}' s_m}(x).$$

□

To prove Conjecture 3.4 for bipartite graphs it remains to prove that

$$\{\det(xI - A_{\mathbf{s}}(G))\}_{\mathbf{s}\in\{-1,1\}^E}$$

is an interlacing family. To do this we will use the following theorem.

Theorem 5.11. *Let $S_1, \ldots, S_m$ be finite nonempty sets and $\{f_{\mathbf{s}}(x)\}_{\mathbf{s}\in S_1\times\cdots\times S_m}$ a set of polynomials of the same degree and with positive leading coefficients.*

If for all choices of independent random variables $\sigma_1 \in S_1, \ldots, \sigma_m \in S_m$ the expected polynomial

$$\mathbb{E}f_\sigma(x)$$

is real–rooted, then $\{f_{\mathbf{s}}(x)\}_{\mathbf{s}\in S_1\times\cdots\times S_m}$ is an interlacing family.

Moreover for each such tuple of random variables there is an $\mathbf{s} \in S_1 \times \cdots \times S_m$ with $\mathbb{P}[\sigma = \mathbf{s}] > 0$ such that the largest zero of $f_{\mathbf{s}}(x)$ is no larger than the largest zero of $\mathbb{E}f_\sigma$.

Proof. Let $s_1 \cdots s_k \in S_1 \times \cdots \times S_k$ be fixed and let $\sigma_1, \ldots, \sigma_m$ be independent random variables defined by

- $\sigma_1, \ldots, \sigma_k$ are deterministic, i.e., $\mathbb{P}[\sigma_i = s_i] = 1$ for all $1 \le i \le k$,
- If $S_{k+1} = \{t_1, \ldots, t_\ell\}$, let $\mathbb{P}[\sigma_{k+1} = t_i] = p_i$ for all $1 \le i \le \ell$.
- σ_j is uniform on S_j for $k + 2 \le j \le m$.

Then

$$\mathbb{E}f_\sigma = |S_{k+2}|^{-1} \cdots |S_m|^{-1} \sum_{j=1}^{\ell} p_j f_{s_1\cdots s_k t_j}$$

is real–rooted by assumption. By Theorem 5.7 $\{f_{s_1\cdots s_k t_j}\}_{j=1}^{\ell}$ have a common interleaver and the first part of the theorem follows.

The second part of the theorem follows by considering the interlacing family $\{g_{\mathbf{s}}\}$ defined by $g_{\mathbf{s}} = \mathbb{P}[\sigma = \mathbf{s}]f_{\mathbf{s}}$. ☐

6. Hyperbolic polynomials

The notion of hyperbolic polynomials is a multivariate generalization of real–rootedness which has its origin in PDE theory where it was studied by Petrovsky, Gårding, Bott, Atiyah and Hörmander. During recent years hyperbolic polynomials have been studied in diverse areas such as control theory, optimization, probability theory, computer science and combinatorics.

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) > 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e}-\mathbf{x})$ has only real zeros. Here are some examples of hyperbolic polynomials:

(1) Let $h(\mathbf{x}) = x_1 \cdots x_n$. Then $h(\mathbf{x})$ is hyperbolic with respect to any vector $\mathbf{e} \in \mathbb{R}^n_{++} = (0,\infty)^n$:

$$h(t\mathbf{e} - \mathbf{x}) = \prod_{j=1}^{n}(te_j - x_j).$$

(2) Let $X = (x_{ij})_{i,j=1}^n$ be a matrix of $n(n + 1)/2$ variables where we impose $x_{ij} = x_{ji}$. Then $\det(X)$ is hyperbolic with respect to $I = \mathrm{diag}(1, \ldots, 1)$. Indeed $t \mapsto \det(tI - X)$ is the characteristic polynomial of the symmetric matrix X, so it has only real zeros. Hence hyperbolic polynomials are

generalizations of determinants, and it is often useful to think about determinants and matrices to get an intuition for the theory of hyperbolic polynomials.

More generally we may consider complex hermitian $Z = (x_{jk}+iy_{jk})_{i,j=1}^n$ (where $i = \sqrt{-1}$) of n^2 real variables where we impose $x_{jk} = x_{kj}$ and $y_{jk} = -y_{kj}$, for all $1 \leq j, k \leq n$. Then $\det(Z)$ is a real polynomial which is hyperbolic with respect to I.

(3) Let $h(\mathbf{x}) = x_1^2 - x_2^2 - \cdots - x_n^2$. Then h is hyperbolic with respect to $(1, 0, \dots, 0)^T$.

A recent celebrated theorem of Helton and Vinnikov says that all hyperbolic polynomials in three variables arise from the determinant:

Theorem 6.1 ([3, 5]). *Suppose that $h(x,y,z)$ is of degree d and hyperbolic with respect to $e = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(e) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1A + e_2B + e_3C = I$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

Suppose h is hyperbolic with respect to $\mathbf{e}$, and of degree d. We may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})), \tag{6.1}$$

where $\lambda_1(\mathbf{x}) \leq \cdots \leq \lambda_d(\mathbf{x})$ are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). In particular

$$h(\mathbf{x}) = \lambda_1(\mathbf{x}) \cdots \lambda_d(\mathbf{x}). \tag{6.2}$$

By homogeneity we have

$$\lambda_j(s\mathbf{x}) = s\lambda_j(\mathbf{x}) \text{ for all } s > 0, \quad \text{and} \quad \lambda_j(\mathbf{x}+s\mathbf{e}) = \lambda_j(\mathbf{x})+s \text{ for all } s \in \mathbb{R}, \tag{6.3}$$

for all $1 \leq j \leq n$, $\mathbf{x} \in \mathbb{R}^n$ and $s \in \mathbb{R}$.

The (open) *hyperbolicity cone* is the set

$$\Lambda_{++} = \Lambda_{++}(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_1(\mathbf{x}) > 0\}.$$

We denote its closure by $\Lambda_+ = \Lambda_+(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_1(\mathbf{x}) \geq 0\}$. Since $h(t\mathbf{e} - \mathbf{e}) = h(\mathbf{e})(t-1)^d$ we see that $\mathbf{e} \in \Lambda_{++}$. The hyperbolicity cones for the examples above are:

(1) $\Lambda_{++}(\mathbf{e}) = \mathbb{R}^n_{++}$.
(2) $\Lambda_{++}(I)$ is the cone of symmetric positive definite matrices.
(3) $\Lambda_{++}(1, 0, \dots, 0)$ is the *Lorentz cone*

$$\left\{ \mathbf{x} \in \mathbb{R}^n : x_1 > \sqrt{x_2^2 + \cdots + x_n^2} \right\}.$$

Proposition 6.2. *The hyperbolicity cone is the connected component of*

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

which contains $\mathbf{e}$.

Proof. Let C be the connected component that contains $\mathbf{e}$. First we prove $C \subseteq \Lambda_{++}$. Suppose that $\mathbf{x}(s)$, $0 \leq s \leq 1$ is a continous path in C connecting $\mathbf{e} = x(0)$ and $\mathbf{x} = \mathbf{x}(1) \in C$. Then $\lambda_1(\mathbf{x}(s)) > 0$ for all $0 \leq s \leq 1$ for otherwise $\lambda_1(\mathbf{x}(s)) = 0$

for some $0 \leq s \leq 1$ which implies $h(\mathbf{x}(s)) = 0$ contrary to the assumption that $\mathbf{x}(s) \in C$.

One the other hand if $\mathbf{x} \in \Lambda_{++}$, then by homogeneity

$$h(t\mathbf{x} + (1-t)\mathbf{e}) = h(\mathbf{e}) \prod_{j=1}^{d} (t\lambda_j(\mathbf{x}) + (1-t)).$$

Since $\lambda_j(\mathbf{x}) > 0$ for all j we see that $t\mathbf{x} + (1-t)\mathbf{e} \in C$ for all $0 \leq t \leq 1$. $\square$

Theorem 6.3. *Suppose h is hyperbolic with respect to $\mathbf{e}$.*

(i) *If $\mathbf{e}' \in \Lambda_{++}(\mathbf{e})$, then h is hyperbolic with respect to $\mathbf{e}'$, and $\Lambda_{++}(\mathbf{e}') = \Lambda_{++}(\mathbf{e})$.*
(ii) *$\Lambda_{++}(\mathbf{e})$ is a convex cone.*

Proof. Suppose $\mathbf{e}' \in \Lambda_{++}(\mathbf{e})$. We claim that for each $\mathbf{y} \in \mathbb{R}^n$ all zeros of the polynomial (in t) $p(\mathbf{y}; t) = h(t\mathbf{e}' - i\mathbf{e} - \mathbf{y})$ have positive imaginary parts. To see that the claim implies that h is hyperbolic with respect to $\mathbf{e}'$ consider the polynomial $\epsilon^d p(\mathbf{y}/\epsilon, t/\epsilon) = h(t\mathbf{e}' - \epsilon i\mathbf{e} - \mathbf{y})$ whose zeros have positive imaginary parts by the claim. By Hurwitz' theorem, the zeros of the real polynomial $t \mapsto h(t\mathbf{e}' - \mathbf{y})$ have nonnegative imaginary parts. Since non-real zeros come in complex conjugate pairs we see that all zeros of $t \mapsto h(t\mathbf{e}' - \mathbf{y})$ are real, and thus h is hyperbolic with respect to $\mathbf{e}'$.

The claim is true for $\mathbf{y} = 0$, since then the zeros are $i\lambda_j(\mathbf{e}')^{-1}$ for $1 \leq j \leq d$. Suppose that the claim fails for some $\mathbf{y} \in \mathbb{R}^n$, and consider the line segment $\{\theta\mathbf{y} : 0 \leq \theta \leq 1\}$. By Hurwitz' theorem, the claim is true for $\theta\mathbf{y}$ for all sufficiently small $\theta \geq 0$. Hence at least one zero of $t \mapsto h(t\mathbf{e}' - i\mathbf{e} - \theta\mathbf{y})$ will cross the real axis for some θ between 0 and 1. For such a θ we have $0 = h(-i\mathbf{e} - \theta\mathbf{y} + \alpha\mathbf{e}')$, for some $\alpha \in \mathbb{R}$. This contradicts the hyperbolicity of h with respect to $\mathbf{e}$ and proves the claim, which then establishes (i).

That $\Lambda_{++}(\mathbf{e}') = \Lambda_{++}(\mathbf{e})$ now follows from Proposition 6.2.

By the proof of Proposition 6.2 we have that for each $\mathbf{x} \in \Lambda_{++}(\mathbf{e})$ the line segment between $\mathbf{e}$ and $\mathbf{x}$ is in $\Lambda_{++}(\mathbf{e})$ whenever $\mathbf{x} \in \Lambda_{++}(\mathbf{e})$. If $\mathbf{x}, \mathbf{y} \in \Lambda_{++}(\mathbf{e})$, then since $\Lambda_{++}(\mathbf{e}) = \Lambda_{++}(\mathbf{x}) = \Lambda_{++}(\mathbf{y})$, the line segment between $\mathbf{x}$ and $\mathbf{y}$ is in $\Lambda_{++}(\mathbf{x}) (= \Lambda_{++}(\mathbf{e}))$. $\square$

Theorem 6.4. *Let $\lambda_1(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ be given by (6.1). Then $\lambda_1(\mathbf{x})$ is concave.*

Proof. By (6.3)

$$\lambda_1(\mathbf{x}) = \max\{s \in \mathbb{R} : \mathbf{x} - s\mathbf{e} \in \Lambda_+\}.$$

Thus $\mathbf{x} - \lambda_1(\mathbf{x})\mathbf{e}, \mathbf{y} - \lambda_1(\mathbf{y})\mathbf{e} \in \Lambda_+$ which by Theorem 6.3 gives $\mathbf{x} + \mathbf{y} - (\lambda_1(\mathbf{x}) + \lambda_1(\mathbf{y}))\mathbf{e} \in \Lambda_+$, and thus $\lambda_1(\mathbf{x} + \mathbf{y}) \geq \lambda_1(\mathbf{x}) + \lambda_1(\mathbf{y})$. $\square$

Lemma 6.5. *Suppose h is hyperbolic with respect to $\mathbf{e}$ and let $\mathbf{v} \in \mathbb{R}^n$. The following are equivalent:*

(1) *All eigenvalues of $\mathbf{v}$ (with respect to $\mathbf{e}$) are zero;*
(2) *$\mathbf{v} \in \Lambda_+$ and $-\mathbf{v} \in \Lambda_+$.*

Proof. It follows by homogeneity that $\lambda_j(-\mathbf{x}) = -\lambda_{d+1-j}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, from which the lemma follows. $\square$

Recall that the *directional derivative* of $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ with respect to $\mathbf{v} = (v_1, \dots, v_n)^T \in \mathbb{R}^n$ is defined as

$$D_{\mathbf{v}} h(\mathbf{x}) = \sum_{k=0}^{n} v_k \frac{\partial h}{\partial x_k}(\mathbf{x}),$$

and note that

$$(D_{\mathbf{v}} h)(\mathbf{x} + t\mathbf{v}) = \frac{d}{dt} h(\mathbf{x} + t\mathbf{v}).$$

Theorem 6.6. *Let $h(\mathbf{x})$ be a hyperbolic polynomial and let $\mathbf{v} \in \Lambda_+$ be such that $D_{\mathbf{v}} h(\mathbf{x}) \not\equiv 0$. Then*

1. *$D_{\mathbf{v}} h(\mathbf{x})$ is hyperbolic with hyperbolicity cone containing Λ_{++}. Moreover, for each $\mathbf{x} \in \mathbb{R}^n$, the polynomial $D_{\mathbf{v}} h(t\mathbf{e} - \mathbf{x})$ is an interleaver of $h(t\mathbf{e} - \mathbf{x})$.*
2. *The polynomial $h(\mathbf{x}) - y D_{\mathbf{v}} h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n, y]$ is hyperbolic with hyperbolicity cone containing $\Lambda_{++} \times \{0\}$.*
3. *The rational function*

$$\mathbf{x} \mapsto \frac{h(\mathbf{x})}{D_{\mathbf{v}} h(\mathbf{x})}$$

is concave on Λ_{++}.

Proof. (2). Suppose first that $\mathbf{v} \in \Lambda_{++}$. Then $h(t\mathbf{v} - \mathbf{x}) + aD_{\mathbf{v}} h(t\mathbf{v} - \mathbf{x}) = (1 + ad/dt)h(t\mathbf{v} - \mathbf{x})$ is real–rooted for all $\mathbf{x} \in \mathbb{R}^n$ and $a \in \mathbb{R}$ by Lemma 5.2. Thus $H_{\mathbf{v}}(\mathbf{x}, y) := h(\mathbf{x}) - y D_{\mathbf{v}} h(\mathbf{x})$ is hyperbolic with respect to $\mathbf{v} \oplus 0$. If $\mathbf{e} \in \Lambda_{++}$, then $H_{\mathbf{v}}(\mathbf{e} \oplus 0) = h(\mathbf{e}) > 0$ which by Proposition 6.2 proves (2) whenever $\mathbf{v} \in \Lambda_{++}$. If $\mathbf{v} \in \Lambda_+$ is on the boundary and $a \in \mathbb{R}$, then by Hurwitz' theorem $p(t) = h(t\mathbf{e} - \mathbf{x}) + aD_{\mathbf{v}} h(t\mathbf{e} - \mathbf{x})$ is real–rooted or identically zero. However, the leading coefficient of $p(t)$ is $h(\mathbf{e}) > 0$, so that $p(t)$ is real–rooted and $H_{\mathbf{v}}(\mathbf{e} \oplus 0) = h(\mathbf{e}) > 0$.

(1) follows immediately from (2) and Proposition 5.8 if we can prove that $D_{\mathbf{v}} h(\mathbf{e}) > 0$ for all $\mathbf{e} \in \Lambda_{++}$. By (6.1), we see that

$$\frac{D_{\mathbf{v}} h(\mathbf{e})}{h(\mathbf{e})} = \sum_{j=1}^{d} \lambda_j(\mathbf{v}),$$

and since $\mathbf{v} \in \Lambda_+$ we have $\lambda_j(\mathbf{v}) \geq 0$ for all $1 \leq j \leq d$. Hence if $D_{\mathbf{v}} h(\mathbf{e}) \leq 0$ for some $\mathbf{e} \in \Lambda_{++}$, then $\lambda_j(\mathbf{v}) = 0$ for all $1 \leq j \leq d$. By Lemma 6.5 this condition does not depend on the choice of $\mathbf{e} \in \Lambda_{++}$. Thus $D_{\mathbf{v}} h(\mathbf{e}') = 0$ for all $\mathbf{e}' \in \Lambda_{++}$. Since Λ_{++} is open and non-empty this implies $D_{\mathbf{v}} h(\mathbf{x}) \equiv 0$ contrary to the assumptions.

(3). If $\mathbf{x} \in \Lambda_{++}$, then (by Proposition 6.2) $\mathbf{x} \oplus y$ is in the closure of the hyperbolicity cone of $H(\mathbf{x}, y)$ if and only if

$$y \leq \frac{h(\mathbf{x})}{D_{\mathbf{v}} h(\mathbf{x})}.$$

Since hyperbolicity cones are convex we have for all $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda_{++}$.

$$y_1 \leq \frac{h(\mathbf{x}_1)}{D_{\mathbf{v}} h(\mathbf{x}_1)} \text{ and } y_2 \leq \frac{h(\mathbf{x}_2)}{D_{\mathbf{v}} h(\mathbf{x}_2)} \text{ imply } y_1 + y_2 \leq \frac{h(\mathbf{x}_1 + \mathbf{x}_2)}{D_{\mathbf{v}} h(\mathbf{x}_1 + \mathbf{x}_2)},$$

from which (3) follows. □

Let $h(\mathbf{x})$ be hyperbolic with respect to $\mathbf{e}$. The *inertia* of $\mathbf{v} \in \mathbb{R}^n$ is the triple $(n_-(\mathbf{v}, \mathbf{e}), n_0(\mathbf{v}, \mathbf{e}), n_+(\mathbf{v}, \mathbf{e}))$, where $n_0(\mathbf{v}, \mathbf{e})$ is the number of eigenvalues of $\mathbf{v}$ (with respect to $\mathbf{e}$) that are equal to zero and $n_\pm(\mathbf{v}, \mathbf{e})$ is the number of positive/negative eigenvalues.

Proposition 6.7. *The inertia does not depend on the choice of* $\mathbf{e} \in \Lambda_{++}$.

Proof. Let $\mathbf{e}, \mathbf{e}' \in \Lambda_{++}$ and $\mathbf{x} \in \mathbb{R}^n$, and let $m(h)$ and $m'(h)$ be the multiplicity of 0 as an eigenvalue of $\mathbf{x}$ with respect to $\mathbf{e}$ and $\mathbf{e}'$, respectively. Let us prove that $m(h) = m'(h)$ by induction over d, the degree of h. By (6.2) we have $m(h) > 0$ if and only if $m'(h) > 0$. Suppose without loss of generality that $m(h) \leq m'(h)$. Then, by Theorem 6.6 (1) $D_{\mathbf{e}}h$ is hyperbolic with respect to $\mathbf{e}$ and $\mathbf{e}'$. Moreover $m(D_{\mathbf{e}}h) = m(h) - 1$ and $m'(D_{\mathbf{e}}h) \geq m'(h) - 1$, since $D_{\mathbf{e}}h(t\mathbf{e}' - \mathbf{x})$ is an interleaver of $h(t\mathbf{e}' - \mathbf{x})$ by Theorem 6.6 (1). By induction we have $m'(D_{\mathbf{e}}h) = m(D_{\mathbf{e}}h)$, and hence $m(h) \geq m'(h)$.

To see that the above proves the proposition assume that the inertia of $\mathbf{x}$ with respect to $\mathbf{e}$ and $\mathbf{e}'$ is $N = (a + r, b, c)$ and $N' = (a, b, c + r)$, respectively (where $r \geq 1$). Consider the parametrized polynomial

$$f_s(t) = t^{-b} h\big(t((1 - s)\mathbf{e} + s\mathbf{e}') - \mathbf{x}\big).$$

As s runs from 0 to 1 exactly r zeros will change from negative to positive. Hence, by Hurwitz' theorem, there is a number $s \in (0, 1)$ such that $f_s(0) = 0$, contrary to what was just proved above. $\square$

We denote the inertia of $\mathbf{v}$ by $(n_-(\mathbf{v}), n_0(\mathbf{v}), n_+(\mathbf{v}))$. The *rank* of $\mathbf{v}$ is defined as $\mathrm{rk}(\mathbf{v}) = n_-(\mathbf{v}) + n_+(\mathbf{v}) = d - n_0(\mathbf{v})$.

Note that for a univariate polynomial $f(t)$, we have

$$\left(\sum_{k=0}^{\infty} \frac{(-y)^k (d/dt)^k}{k!}\right) f(t) = f(t - y)$$

and hence

$$h(\mathbf{x} - y\mathbf{v}) = \left(\sum_{k=0}^{\infty} \frac{(-y)^k D_{\mathbf{v}}^k}{k!}\right) h(\mathbf{x}). \tag{6.4}$$

Thus

$$h(\mathbf{e} - t\mathbf{v}) = h(\mathbf{e}) \prod_{j=1}^{d} (1 - t\lambda_j(\mathbf{v})) = \sum_{k=0}^{d} (-1)^k \frac{D_{\mathbf{v}}^k h(\mathbf{e})}{k!} t^k,$$

and hence

$$\mathrm{rk}(\mathbf{v}) = \deg h(\mathbf{e} - t\mathbf{v}) = \min\{k : D_{\mathbf{v}}^j h(\mathbf{e}) = 0 \text{ for all } j > k\}.$$

By Proposition 6.7, the rank does not depend on the choice of $\mathbf{e} \in \Lambda_{++}$. Hence if $D_{\mathbf{v}}^{k+1} h(\mathbf{e}) = D_{\mathbf{v}}^{k+2} h(\mathbf{e}) = \cdots = 0$ for some $\mathbf{e} \in \Lambda_{++}$, then $D_{\mathbf{v}}^{k+1} h(\mathbf{e}') = D_{\mathbf{v}}^{k+2} h(\mathbf{e}') = \cdots = 0$ for all $\mathbf{e}' \in \Lambda_{++}$. Since Λ_{++} has non-empty interior this means that $D_{\mathbf{v}}^k h \equiv 0$. We have thus the following equivalent definition of rank

$$\mathrm{rk}(\mathbf{v}) = \min\{k : D_{\mathbf{v}}^{k+1} h \equiv 0\}, \tag{6.5}$$

which makes sense for polynomials which are not necessarily hyperbolic.

7. Mixed hyperbolic polynomials

If $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ let $h[\mathbf{v}_1, \ldots, \mathbf{v}_m]$ be the polynomial in $\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ defined by

$$h[\mathbf{v}_1, \ldots, \mathbf{v}_m] = \prod_{j=1}^{m} \left(1 - y_j D_{\mathbf{v}_j}\right) h(\mathbf{x}).$$

By iterating Theorem 6.6 (2) we get:

Theorem 7.1. *If $h(\mathbf{x})$ is hyperbolic with hyperbolicity cone Λ_{++} and $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Lambda_+$, then $h[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is hyperbolic with hyperbolicity cone containing $\Lambda_{++} \times \{0\}$.*

Lemma 7.2. *If $\mathbf{v}_1, \dots, \mathbf{v}_m$ have rank at most one, then*

$$h[\mathbf{v}_1, \dots, \mathbf{v}_m] = h(\mathbf{x} - y_1\mathbf{v}_1 - \cdots - y_m\mathbf{v}_m).$$

Proof. If $\mathbf{v}$ has rank at most one, then $D_{\mathbf{v}}^k h \equiv 0$ for all $k \geq 2$. Hence, by (6.4),

$$h(\mathbf{x} - y\mathbf{v}) = \left(\sum_{k=0}^{\infty} \frac{(-y)^k D_{\mathbf{v}}^k}{k!}\right) h(\mathbf{x}) = (1 - yD_{\mathbf{v}})h(\mathbf{x}),$$

from which the lemma follows. □

Note that $h[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is affine in each coordinate, i.e., for all $p \in \mathbb{R}$:

$$h[p\mathbf{v}_1 + (1-p)\mathbf{u}_1, \dots, v_m] = ph[\mathbf{v}_1, \dots, \mathbf{v}_m] + (1-p)h[\mathbf{u}_1, \dots, \mathbf{v}_m].$$

Hence if $\mathbf{v}_1, \dots, \mathbf{v}_m$ are independent random variables, then

$$\mathbb{E}h[\mathbf{v}_1, \dots, \mathbf{v}_m] = h[\mathbb{E}\mathbf{v}_1, \dots, \mathbb{E}\mathbf{v}_m]. \tag{7.1}$$

Theorem 7.3. *Let $h(\mathbf{x})$ be a hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$. Let $V_1, \dots, V_m$ be finite sets of vectors in Λ_+ and let $\mathbf{w} \in \mathbb{R}^{n+m}$. For $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_m) \in V_1 \times \cdots \times V_m$, let*

$$f_{\mathbf{V}}(x) := h[\mathbf{v}_1, \dots, \mathbf{v}_m](x\mathbf{e} + \mathbf{w}).$$

Then $\{f_{\mathbf{V}}(x)\}_{\mathbf{V} \in V_1 \times \cdots \times V_m}$ is an interlacing family.

In particular if in addition all vectors in $V_1 \cup \cdots \cup V_m$ have rank at most one, and

$$g_{\mathbf{V}}(x) := h(x\mathbf{e} + \mathbf{w} - \mathbf{v}_1 - \cdots - \mathbf{v}_m),$$

where $\mathbf{w} \in \mathbb{R}^n$, then $\{g_{\mathbf{V}}(x)\}_{\mathbf{V} \in V_1 \times \cdots \times V_m}$ is an interlacing family.

Proof. Let $\mathbf{v}_1 \in V_1, \dots, \mathbf{v}_m \in V_m$ be independent random variables. Then the polynomial $\mathbb{E}h[\mathbf{v}_1, \dots, \mathbf{v}_m] = h[\mathbb{E}\mathbf{v}_1, \dots, \mathbb{E}\mathbf{v}_m]$ is hyperbolic with respect to $\mathbf{e} \oplus 0$ by Theorem 7.1 (since $\mathbb{E}\mathbf{v}_i \in \Lambda_+$ for all i by convexity). The theorem now follows from Theorem 5.11.

□

The following corollary establishes Conjecture 3.4 for bipartite graphs.

Corollary 7.4. *Let $G = (V, E)$ be a graph. Then*

$$\{\det(xI - A_{\mathbf{s}}(G))\}_{\mathbf{s} \in \{-1,1\}^E}$$

is an interlacing family.

Proof. For $e = \{i, j\} \in E$, let $B_e(\pm 1) = \pm E_{ij} + E_{ii} + E_{jj}$ and $V_e = \{B_e(-1), B_e(1)\}$. Then $B_e(\pm 1)$ is positive semidefinite of rank one, and hence $B_e(\pm 1)$ is in the closure of the hyperbolicity cone of det. Moreover if $\mathbf{s} \in \{-1, 1\}^E$ then

$$\det\left(xI + D(G) - \sum_{e \in E} B_e(s(e))\right) = \det(xI - A_{\mathbf{s}}(G)),$$

where $D(G)$ is the diagonal matrix with (i, i)-entry equal to the degree of i. Hence the corollary follows from Theorem 7.3 □

8. The Kadison–Singer problem

The Kadison–Singer problem is a problem formulated by Kadison and Singer [4] in 1959 and originates in speculations made by Dirac.

Problem 1 (Kadison–Singer)**.** *Does every pure state on the algebra of bounded diagonal operators on the complex Banach space ℓ_2 have a unique extension to a state on the algebra of all bounded operators on ℓ_2?*

This problem was one of the central open problems in operator theory until its recent resolution by Marcus, Spielman and Srivastava [7]. Several equivalent problems have been stated in different mathematical context by Andersson, Akemann, Weaver and others, see ?? for an introduction and more references to Problem 1.

The following conjecture by Weaver [8] is known to imply a positive solution to the Kadison–Singer problem.

Conjecture 8.1. *There are universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds. Let $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathbb{C}^n$ be such that $\|\mathbf{w}_i\| \leq 1$ for all $1 \leq i \leq m$ and*

$$\sum_{i=1}^{m} |\langle \mathbf{u}, \mathbf{w}_i \rangle|^2 = \eta, \tag{8.1}$$

for every unit vector $\mathbf{u} \in \mathbb{C}^n$.

Then there is a partition $S_1 \cup S_2 = \{1, \dots, m\}$ such that

$$\sum_{i \in S_j} |\langle \mathbf{u}, \mathbf{w}_i \rangle|^2 \leq \eta - \theta, \tag{8.2}$$

for every unit vector $\mathbf{u} \in \mathbb{C}^n$ and each $j \in \{1, 2\}$.

Let us formulate a stronger conjecture in terms of hyperbolic polynomials. To do this we formulate the terms used in Conjecture 8.1 in terms that make sense for hyperbolic polynomials.

Let A^* denote the complex transpose of a matrix A. Note that for any $\mathbf{u}, \mathbf{w}_1, \dots, \mathbf{w}_m \in \mathbb{C}^n$

$$\sum_{i=1}^{m} |\langle \mathbf{u}, \mathbf{w}_i \rangle|^2 = \mathbf{u}^* \left(\sum_{i=1}^{m} \mathbf{w}_i \mathbf{w}_i^* \right) \mathbf{u} = \langle A\mathbf{u}, \mathbf{u} \rangle,$$

where $A = \sum_{i=1}^{m} \mathbf{w}_i \mathbf{w}_i^*$. It follows that (8.1) holds if and only if $A = \eta I$, where I is the identity matrix. Since $B_j := \sum_{i \in S_j} \mathbf{w}_i \mathbf{w}_i^*$ we see that (8.2) holds if and only if

$$\lambda_n(B_j) \leq \eta - \theta,$$

for $j \in \{1, 2\}$, where $\lambda_n(B_j)$ is the largest eigenvalue of B_j.

Recall that a hermitian $n \times n$ matrix A has rank at most one if and only if $A = \mathbf{u}\mathbf{u}^*$ for some $\mathbf{u} \in \mathbb{C}^n$. Moreover

$$\lambda_{\max}(A) = \mathrm{Tr}(A) = \|\mathbf{u}\|^2.$$

If h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$, then the *trace* of $\mathbf{v}$ (with respect to $\mathbf{e}$) is defined by

$$\mathrm{Tr}(\mathbf{v}) = \sum_{i=1}^{d} \lambda_i(\mathbf{v}) = \frac{D_{\mathbf{v}} h(\mathbf{e})}{h(\mathbf{e})},$$

where $h(t\mathbf{e}-\mathbf{v}) = h(\mathbf{e})\prod_{i=1}^d (t-\lambda_i(\mathbf{v}))$. Hence $\mathrm{Tr}(\mathbf{v})$ is linear in $\mathbf{v}$. Now it is plain to see that the following theorem implies Conjecture 8.1. Indeed let $\mathbf{u}_i = \mathbf{w}_i\mathbf{w}_i^*/\eta$ and h be the determinant on $n \times n$ complex hermitian matrices.

Theorem 8.2. *There are universal constants $\eta \geq 2$ and $\theta > 0$ such that the following holds. Suppose h is hyperbolic with respect to $\mathbf{e}$ and of degree d. Let $\mathbf{u}_1, \dots, \mathbf{u}_m \in \Lambda_+$ be of rank at most one and such that $\mathrm{Tr}(\mathbf{u}_i) \leq 1/\eta$ for all $1 \leq i \leq m$ and*

$$\sum_{i=1}^m \mathbf{u}_i = \mathbf{e}. \tag{8.3}$$

Then there is a partition $S_1 \cup S_2 = \{1, \dots, m\}$ such that

$$\lambda_{\max}\left(\sum_{i \in S_j} \mathbf{u}_i\right) \leq 1 - \theta/\eta, \tag{8.4}$$

for each $j \in \{1,2\}$.

To prove this we use the following theorem:

Theorem 8.3. *Suppose h is hyperbolic with respect to $\mathbf{e}$. Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be independent random vectors in Λ_+ of rank at most one and with finite supports such that*

$$\mathbb{E}\sum_{i=1}^m \mathbf{v}_i = \mathbf{e}, \tag{8.5}$$

and

$$\mathrm{Tr}(\mathbb{E}\mathbf{v}_i) \leq \epsilon \text{ for all } 1 \leq i \leq m. \tag{8.6}$$

Then

$$\mathbb{P}\left[\lambda_{\max}\left(\sum_{i=1}^m \mathbf{v}_i\right) \leq (1+\sqrt{\epsilon})^2\right] > 0. \tag{8.7}$$

Theorem 8.3 implies the following proposition.

Proposition 8.4. *Suppose h is hyperbolic with respect to $\mathbf{e}$ and of degree d. Let $\mathbf{u}_1, \dots, \mathbf{u}_m \in \Lambda_+(\mathbf{e})$ be of rank at most one and such that $\mathrm{Tr}(\mathbf{u}_i) \leq \alpha$ for all $1 \leq i \leq m$, and $\sum_{i=1}^m \mathbf{u}_i = \mathbf{e}$. Then there exists a partition $S_1 \cup S_2 = \{1, \dots, m\}$ such that*

$$\lambda_{\max}\left(\sum_{i \in S_j} \mathbf{u}_i\right) \leq \frac{(1+\sqrt{2\alpha})^2}{2}, \tag{8.8}$$

for each $j \in \{1,2\}$.

Proof. Consider the hyperbolic polynomial

$$H(\mathbf{x}, \mathbf{x}') = h(\mathbf{x})h(\mathbf{x}') \in \mathbb{R}[x_1, \dots, x_n, x_1', \dots, x_n'],$$

which is hyperbolic with respect to $\mathbf{e} \oplus \mathbf{e}'$, where $\mathbf{e}'$ is a copy of $\mathbf{e}$ in the x_i'-variables. Let $\mathbf{v}_1, \dots \mathbf{v}_m$ be independent random vectors in $\Lambda_+(\mathbf{e}) \oplus \Lambda_+(\mathbf{e}')$ such that

$$\mathbb{P}[\mathbf{v}_i = 2\mathbf{u}_i] = 1/2 \text{ and } \mathbb{P}[\mathbf{v}_i = 2\mathbf{u}_i'] = 1/2,$$

where $\mathbf{u}'_1, \dots, \mathbf{u}'_m$ are the copies in $\Lambda_+(\mathbf{e}')$ of $\mathbf{u}_1, \dots, \mathbf{u}_m$. Then $\mathbb{E}\mathbf{v}_i = \mathbf{u}_i \oplus \mathbf{u}'_i$ and $\mathrm{Tr}(\mathbb{E}\mathbf{v}_i) \le 2\alpha$, and hence

$$\mathbb{E}\sum_{i=1}^m \mathbf{v}_i = \mathbf{e} \oplus \mathbf{e}'.$$

By Theorem 8.3 there is a $T \subseteq \{1, \dots, m\}$ such that

$$\lambda_{\max}\left(\sum_{i\in T} 2\mathbf{u}_i + \sum_{i\notin T} 2\mathbf{u}'_i\right) = \max\left\{\lambda_{\max}\left(\sum_{i\in T} 2\mathbf{u}_i\right), \lambda_{\max}\left(\sum_{i\notin T} 2\mathbf{u}_i\right)\right\} \le (1+\sqrt{2\alpha})^2,$$

and the proposition follows. $\square$

Now let $\alpha = 1/\eta$ in Proposition 8.4. Then

$$\frac{(1+\sqrt{2\alpha})^2}{2} = \frac{1}{2} + \sqrt{\frac{2}{\eta}} + \frac{1}{\eta} \le 1 - \frac{\theta}{\eta}$$

if and only if

$$\theta \le \frac{\eta}{2} - 2\sqrt{\frac{\eta}{2}} - 1.$$

Hence we have a "solution" to Theorem 8.2 (and thus Conjecture 8.1) for all

$$\eta > 6 + 4\sqrt{2} \quad \text{and} \quad 0 < \theta \le \frac{\eta}{2} - 2\sqrt{\frac{\eta}{2}} - 1.$$

For example $\eta = 18$ and $\theta = 2$. It remains to prove Theorem 8.3.

9. Bounds on zeros of mixed hyperbolic polynomials

To prove Theorem 8.3, using the method of interlacing families, we want to bound the zeros of the *mixed characteristic polynomial*

$$t \mapsto h[\mathbf{v}_1, \dots, \mathbf{v}_m](t\mathbf{e} + \mathbf{1}), \tag{9.1}$$

where h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$, $\mathbf{1} \in \mathbb{R}^m$ is the all ones vector, and $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Lambda_+(\mathbf{e})$ satisfies $\mathbf{v}_1 + \cdots + \mathbf{v}_m = \mathbf{e}$ and $\mathrm{Tr}(\mathbf{v}_i) \le \epsilon$ for all $1 \le i \le m$. Note that a real number ρ is larger than the maximum zero of (9.1) if and only if $\rho\mathbf{e} + \mathbf{1}$ is in the hyperbolicity cone of $h[\mathbf{v}_1, \dots, \mathbf{v}_m]$.

For the remainder of this section, let $h \in \mathbb{R}[x_1, \dots, x_n]$ be hyperbolic with respect to $\mathbf{e}$, and let $\mathbf{v}_1, \dots, \mathbf{v}_m \in \Lambda_+$. Let $\partial_j = D_{\mathbf{v}_j}$ and

$$\xi_j[g] = \frac{g}{\partial_j g}.$$

Note that a continuously differentiable concave function $f : (0, \infty) \to \mathbb{R}$ satisfies

$$f(t+\delta) \ge f(t) + \delta f'(t+\delta), \qquad \text{for all } \delta \ge 0.$$

Hence by Theorem 6.6

$$\xi_i[h](\mathbf{x} + \delta\mathbf{v}_j) \ge \xi_i[h](\mathbf{x}) + \delta\partial_j\xi_i[h](\mathbf{x} + \delta\mathbf{v}_j) \tag{9.2}$$

for all $\mathbf{x} \in \Lambda_{++}$. The following elementary identity is left for the reader to verify.

Lemma 9.1.

$$\xi_i[h - \partial_j h] = \xi_i[h] - \frac{\partial_j\xi_i[h]}{1 - \xi_j[\partial_i h]^{-1}}.$$

Lemma 9.2. *If* $\mathbf{x} \in \Lambda_{++}$, $\delta > 1$ *and*

$$\xi_j[h](\mathbf{x}) \geq \frac{\delta}{\delta - 1},$$

then

$$\xi_i[h - \partial_j h](\mathbf{x} + \delta \mathbf{v}_j) \geq \xi_i[h](\mathbf{x}).$$

Proof. Since $\xi_i[h]$ is concave on Λ_{++} and homogeneous of degree one we have

$$\frac{\xi_i[h](\mathbf{x} + \delta \mathbf{v}_j) - \xi_i[h](\mathbf{x})}{\delta} \geq \xi_i[h](\mathbf{v}_j), \qquad \text{for all } \mathbf{x} \in \Lambda_{++}.$$

By letting $\delta \to 0$ we see that

$$\partial_j \xi_i[h](\mathbf{x}) \geq \xi_i[h](\mathbf{v}_j) \geq 0, \qquad \text{for all } \mathbf{x} \in \Lambda_{++}. \tag{9.3}$$

If $\mathbf{x} \in \Lambda_{++}$, then (by Proposition 6.2) $(\mathbf{x}, t)$ is in the closure of the hyperbolicity cone of $h - y\partial_j h$ if and only if $t \leq \xi_j[h](\mathbf{x})$. By Theorem 6.6 the polynomial

$$D_{(\mathbf{v}_i,0)}(h - y\partial_j h) = \partial_i h - y \partial_j \partial_i h$$

is hyperbolic with hyperbolicity cone containing the hyperbolicity cone of $h - y\partial_j h$. Hence if $\mathbf{x} \in \Lambda_{++}$ and $t \leq \xi_j[h](\mathbf{x})$, then $t \leq \xi_j[\partial_i h](\mathbf{x})$, and thus

$$\xi_j[\partial_i h](\mathbf{x}) \geq \xi_j[h](\mathbf{x}), \qquad \text{for all } \mathbf{x} \in \Lambda_{++}. \tag{9.4}$$

By Lemma 9.1 and (9.2)

$$\begin{aligned}
\xi_i[h - \partial_j h](\mathbf{x} + \delta \mathbf{v}_j) - \xi_i[h](\mathbf{x}) &= \xi_i[h](\mathbf{x} + \delta \mathbf{v}_j) - \xi_i[h](\mathbf{x}) - \frac{\partial_j \xi_i[h]}{1 - \xi_j[\partial_i h]^{-1}}(\mathbf{x} + \delta \mathbf{v}_j) \\
&\geq \partial_j \xi_i[h](\mathbf{x} + \delta \mathbf{v}_j) \left(\delta - \frac{\xi_j[\partial_i h](\mathbf{x} + \delta \mathbf{v}_j)}{\xi_j[\partial_i h](\mathbf{x} + \delta \mathbf{v}_j) - 1} \right) \\
&\geq \xi_i[h](\mathbf{v}_j) \left(\delta - \frac{\delta/(\delta - 1)}{\delta/(\delta-1) - 1} \right) = 0,
\end{aligned}$$

where the last inequality follows from (9.3), (9.4) and the concavity of $\xi_j[h]$. □

Consider $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$ and let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard bases of $\mathbb{R}^m$ (inside $\mathbb{R}^n \oplus \mathbb{R}^m$).

Corollary 9.3. *Suppose* h *is hyperbolic with respect to* $\mathbf{e} \in \mathbb{R}^n$, *and let* Γ_+ *be the closure of the hyperbolicity cone of* $h[\mathbf{v}_1, \ldots, \mathbf{v}_m]$, *where* $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{x} \in \Lambda_{++}(\mathbf{e})$. *Suppose* $t_i, t_j > 1$ *are such that*

$$\mathbf{x} + t_k \mathbf{e}_k \in \Gamma_+, \qquad \text{for } k \in \{i, j\}.$$

Then

$$\mathbf{x} + \frac{t_j}{t_j - 1}\mathbf{v}_j + \mathbf{e}_j + t_i \mathbf{e}_i \in \Gamma_+.$$

Moreover if $\mathbf{e} + t_k \mathbf{e}_k \in \Gamma_+$ *for all* $k \in [m]$, *then*

$$\mathbf{x} + \left(1 - \frac{1}{m}\right) \sum_{i=1}^m \frac{t_i}{t_i - 1}\mathbf{v}_i + \left(1 - \frac{1}{m}\right) \sum_{i=1}^m \mathbf{e}_i + \frac{1}{m} \sum_{i=1}^m t_i \mathbf{e}_i \in \Gamma_+.$$

Proof. Let $\delta_k = t_k/(t_k - 1)$. Then

$$\mathbf{x} + t_k\mathbf{e}_k \in \Gamma_+ \text{ if and only if } \xi_k[h] \geq \frac{\delta_k}{\delta_k - 1}.$$

Also $\xi_i[h - \partial_j h](\mathbf{x} + \delta_j \mathbf{v}_j) \geq \delta_i/(\delta_i - 1)$ is equivalent to

$$\mathbf{x} + \delta_j\mathbf{v}_j + \mathbf{e}_j + \frac{\delta_i}{\delta_i - 1}\mathbf{e}_i \in \Gamma_+.$$

Hence the first part follows from Lemma 9.2.

Suppose $\mathbf{e} + t_k\mathbf{e}_k \in \Gamma_+$ for all $k \in [m]$. By the first part we have $\mathbf{x}' + t_2\mathbf{e}_2, \mathbf{x}' + t_3\mathbf{e}_3 \in \Gamma_+$, where

$$\mathbf{x}' = \mathbf{x} + \frac{t_1}{t_1 - 1}\mathbf{v}_1 + \mathbf{e}_1$$

is in the hyperbolicity cone of $(1 - y_1 D_{\mathbf{v}_1})h$. Hence we may apply the first part of the theorem with h replaced by $(1 - y_1 D_{\mathbf{v}_1})h$ to conclude

$$\mathbf{x}' + \frac{t_2}{t_2 - 1}\mathbf{v}_2 + \mathbf{e}_2 + t_3\mathbf{e}_3 = \mathbf{x} + \frac{t_1}{t_1 - 1}\mathbf{v}_1 + \frac{t_2}{t_2 - 1}\mathbf{v}_2 + \mathbf{e}_1 + \mathbf{e}_2 + t_3\mathbf{e}_3 \in \Gamma_+.$$

By continuing this procedure with different orderings we may conclude that

$$\mathbf{x} + \left(\sum_{i=1}^m \frac{t_i}{t_i - 1}\mathbf{v}_i\right) - \frac{t_j}{t_j - 1}\mathbf{v}_j + \left(\sum_{i=1}^m \mathbf{e}_i\right) - \mathbf{e}_j + t_i\mathbf{e}_i \in \Gamma_+,$$

for each $1 \leq j \leq m$. The second part now follows from convexity of Γ_+ upon taking the convex sum of these vectors. $\square$

Theorem 9.4. *Suppose h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \Lambda_+(\mathbf{e})$ be such that $\mathrm{Tr}(\mathbf{v}_j) \leq \epsilon$ for all $1 \leq j \leq m$ and $\mathbf{e} = \mathbf{v}_1 + \cdots + \mathbf{v}_m$. Then the largest root of*

$$\prod_{j=1}^m \left(1 - D_{\mathbf{v}_j}\right) h(t\mathbf{e})$$

is at most $m\epsilon$ if $m\epsilon < 1$ and at most

$$\epsilon + 1 - \frac{1}{m} + 2\left(1 - \frac{1}{m}\right)\left(\sqrt{\epsilon - \frac{1}{m}\left(1 - \frac{1}{m}\right)} - \frac{1}{m}\right) \leq \left(1 + \sqrt{\epsilon - \frac{1}{m}}\right)^2,$$

if $m\epsilon \geq 1$.

Proof. Let $\mathbf{x} = s\mathbf{e}$, where $s > 0$ and $t_i = t$ for $1 \leq i \leq m$ and apply Corollary 9.3. Then for $t > 1$ and $s/t \leq \epsilon$

$$\frac{s + \left(1 - \frac{1}{m}\right)\frac{t}{t-1}}{1 - \frac{1}{m} + \frac{t}{m}}\mathbf{e} + \mathbf{1} \in \Gamma_+.$$

Hence (set $s = t\epsilon$) the maximal root is no larger than

$$\inf\left\{\frac{\epsilon t + \left(1 - \frac{1}{m}\right)\frac{t}{t-1}}{1 - \frac{1}{m} + \frac{t}{m}} : t > 1\right\}.$$

It is a simple exercise to deduce that the infimum is exactly what is displayed in the statement of the theorem. $\square$

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