# Bootstrap percolation on $G_{n,p}$

Thomas Vallier
joint work with Svante Janson, Tomasz Łuczak, Tatyana Turova

Centre for Mathematical Sciences
Helsinki, Finland

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1 Bootstrap percolation...
- ... on a grid
- ... on the Erdös-Rényi random graph $G(n, p)$

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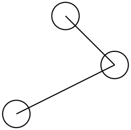

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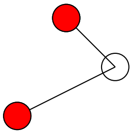

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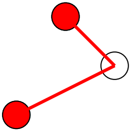

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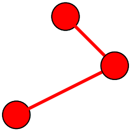

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Dimension $d = 2$, number of necessary incoming activation $k = 2$.

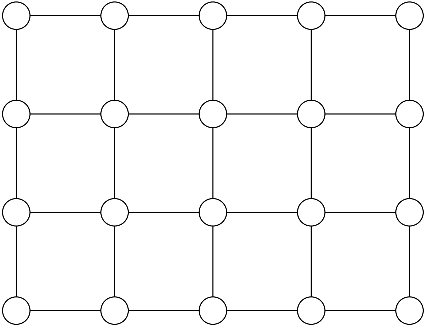

Figure: Consider an $L \times L$ square with sites initially independently declared active with probability $q$.

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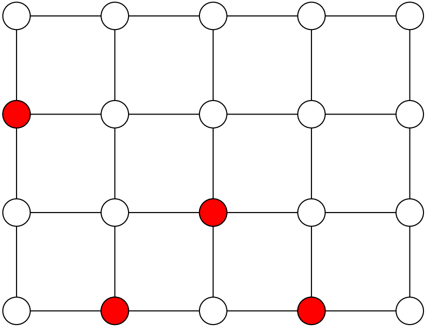

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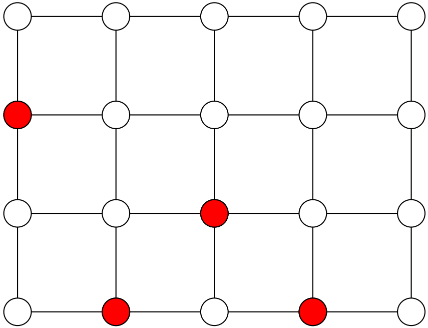

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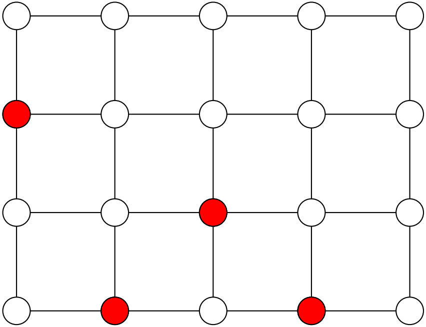

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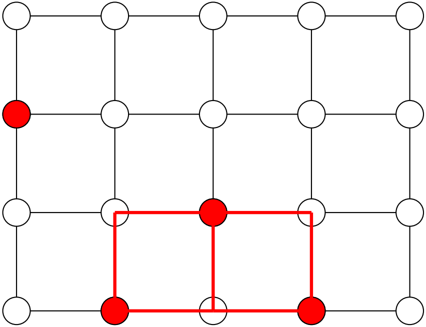

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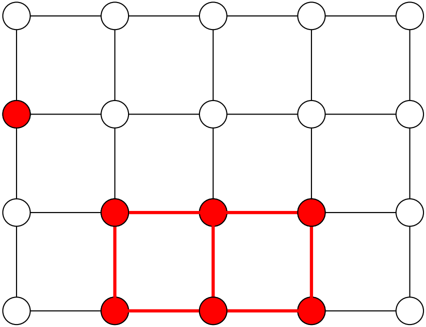

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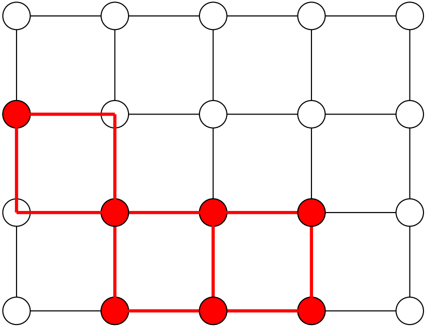

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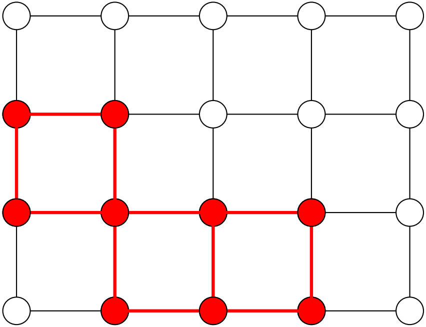

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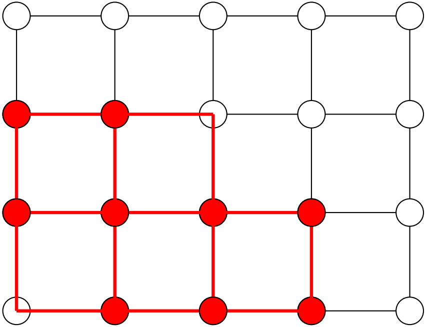

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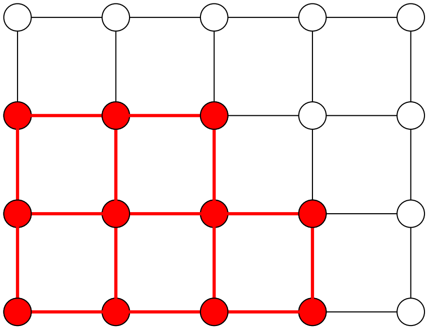

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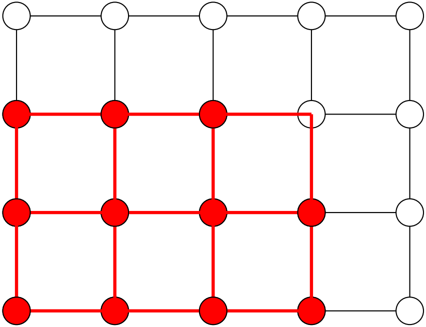

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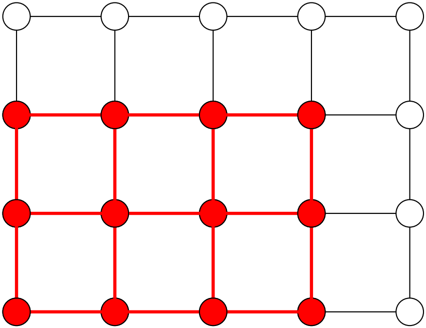

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**Theorem ($d \geq k \geq 2$: Balogh, Bollobás, Duminil-Copin, Morris (2012))**

$$q_c([L]^d, k) = \left(\frac{\lambda(d,k) + o(1)}{\log_{(k-1)}(L)}\right)^{d-k+1}$$

*where* $\log_{(r)}(n) = \log\left(\log_{(r-1)}(n)\right)$

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- A. E. Holroyd, Sharp metastability threshold for two-dimensional bootstrap percolation. *Prob. Theo. Rel. Fields* (2003)
- J. Balogh, B. Bollobás, R. Morris, Bootstrap percolation in three dimensions. *Ann. Prob.* (2009)
- J. Balogh, B. Bollobás, H. Duminil-Copin, R. Morris, The sharp threshold for bootstrap percolation in all dimensions. *Trans. Amer. Math. Soc.* (2012)


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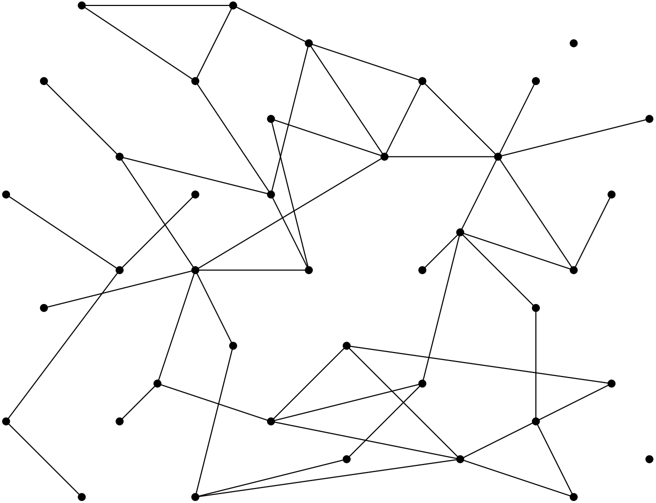

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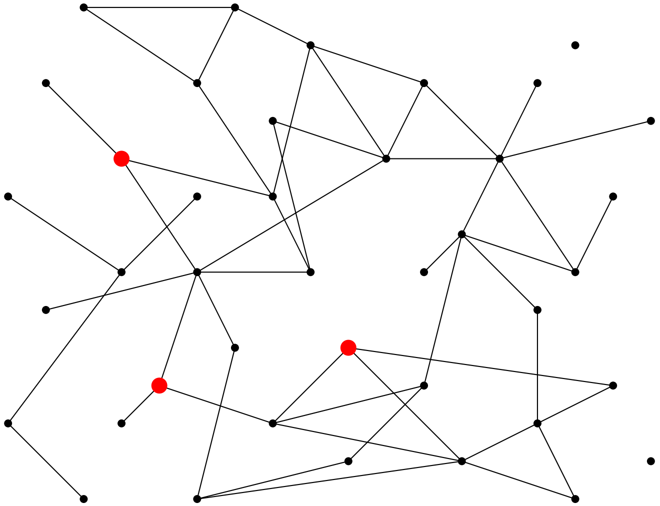

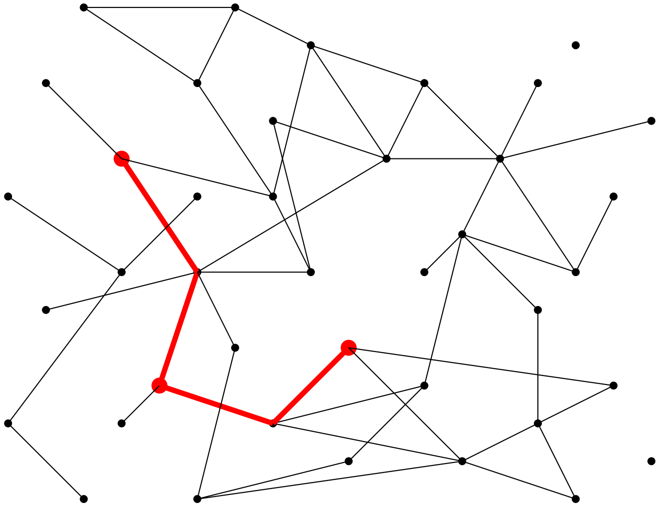

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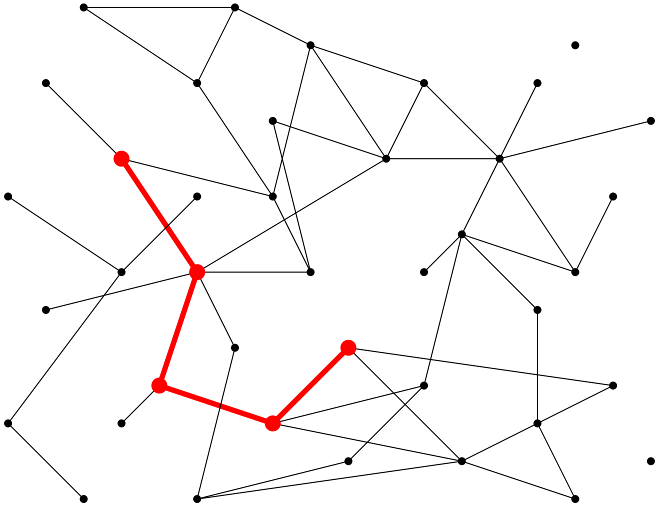

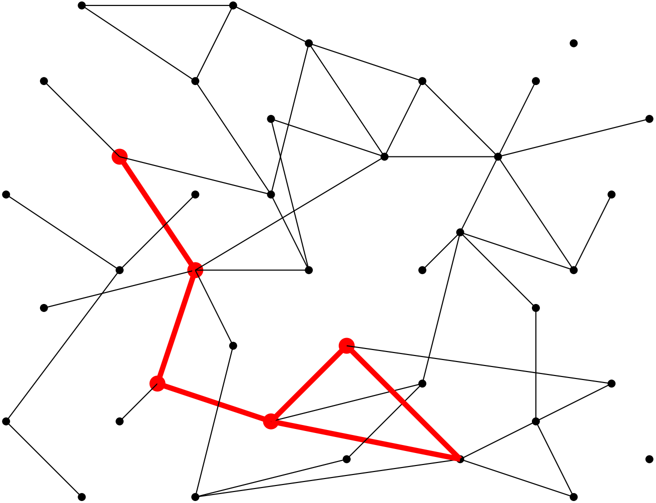

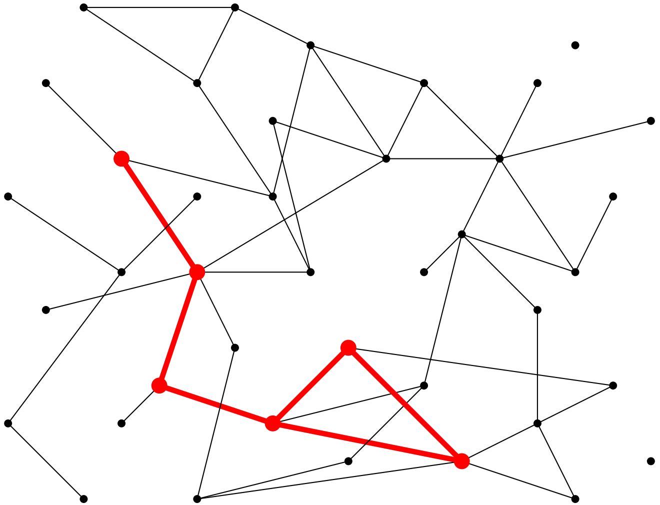

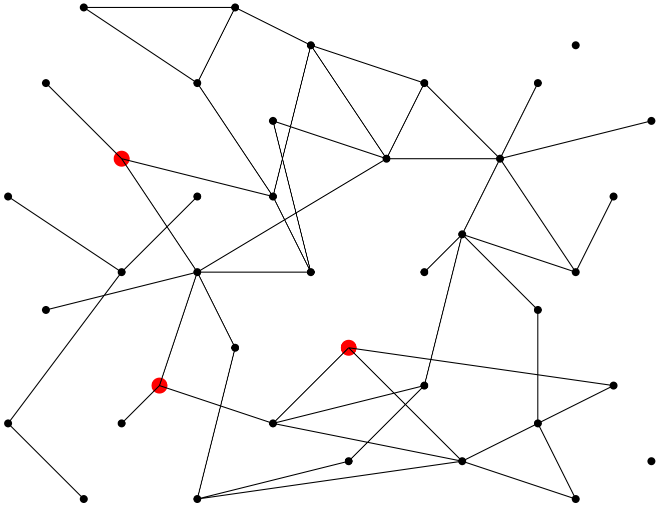

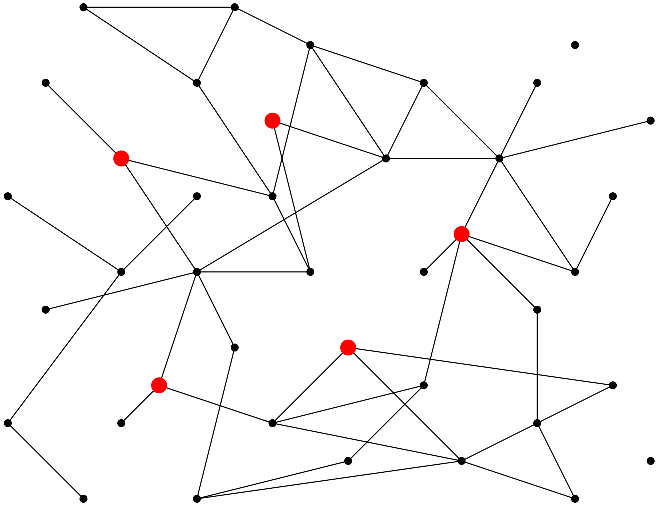

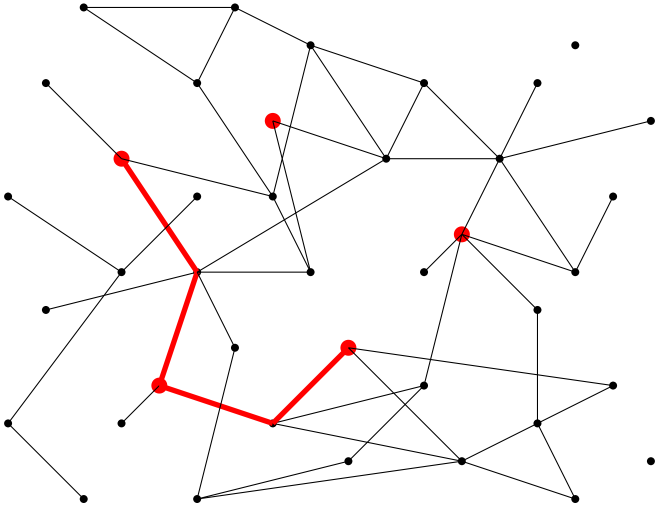

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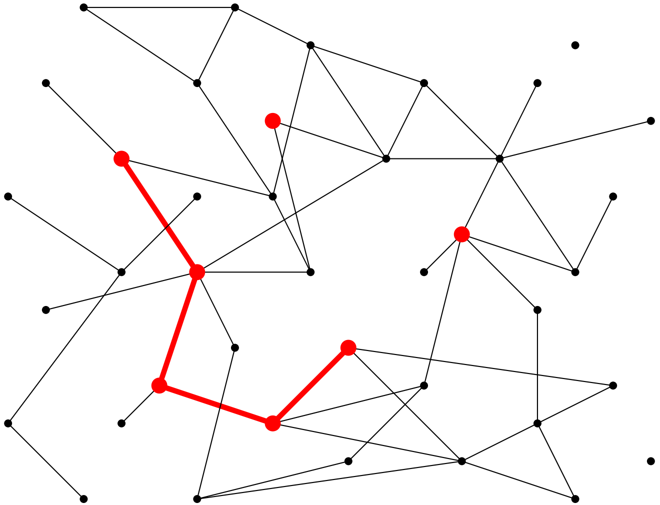

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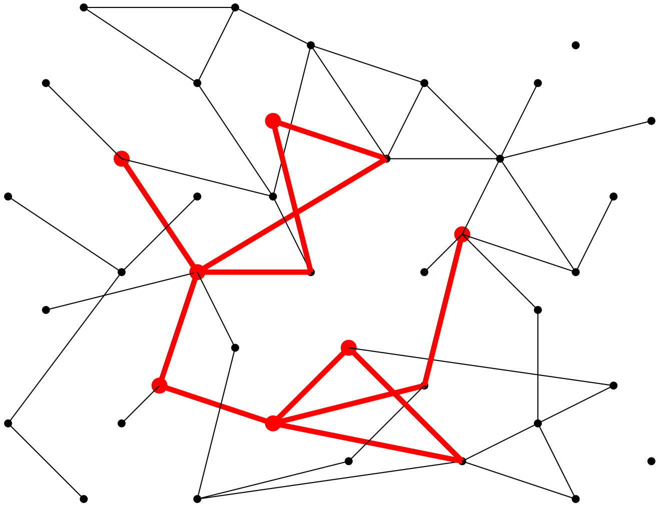

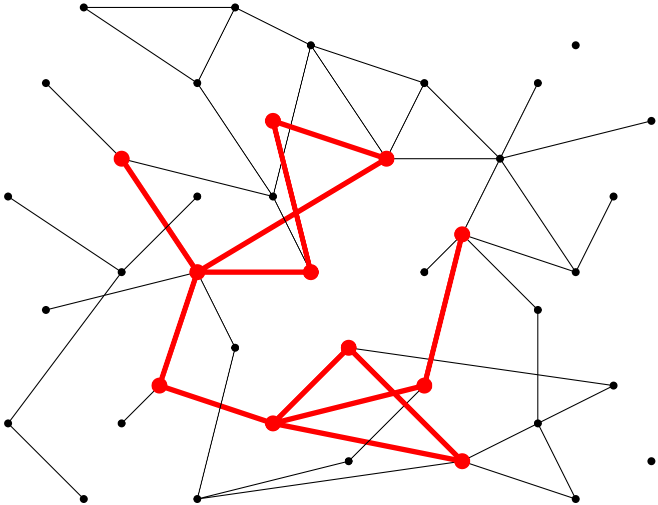

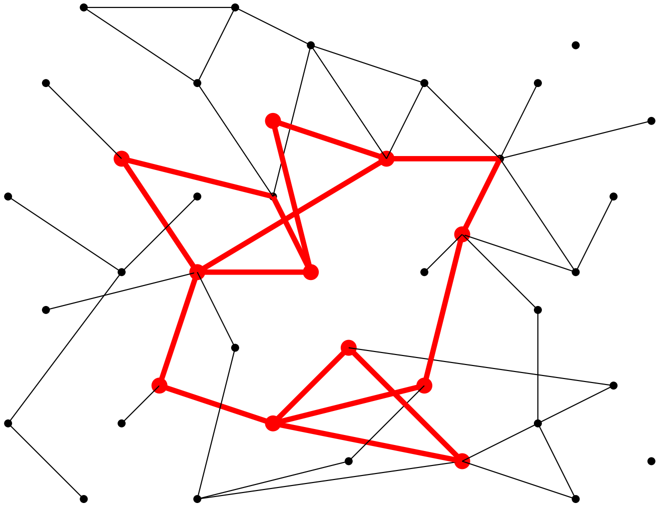

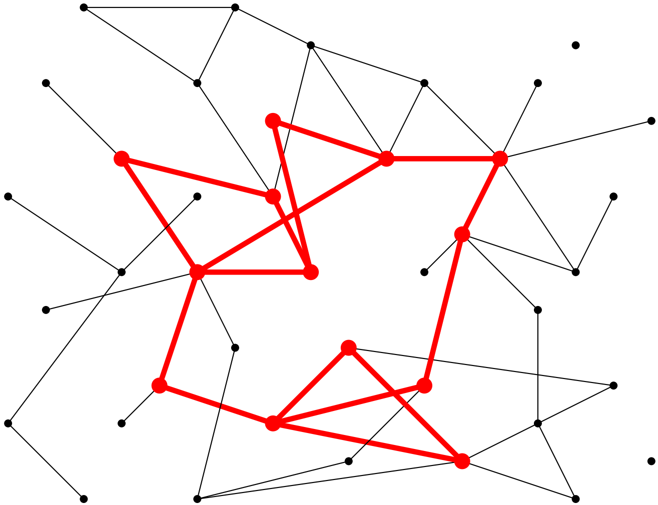

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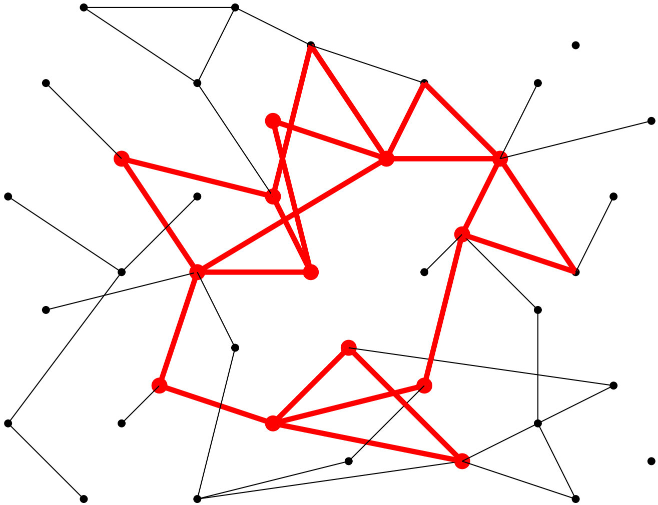

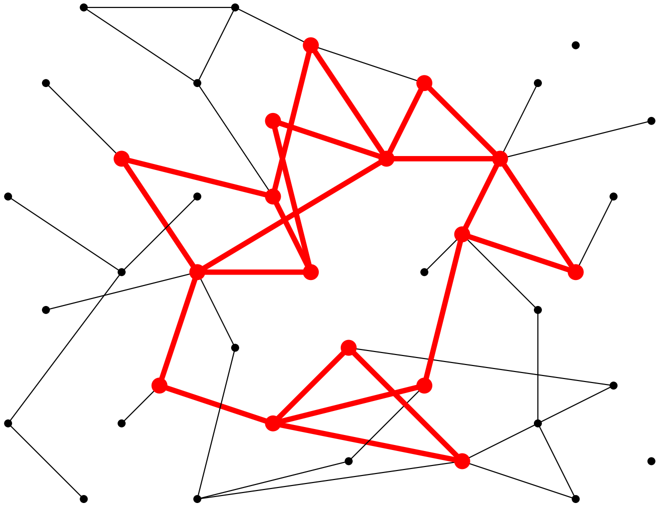

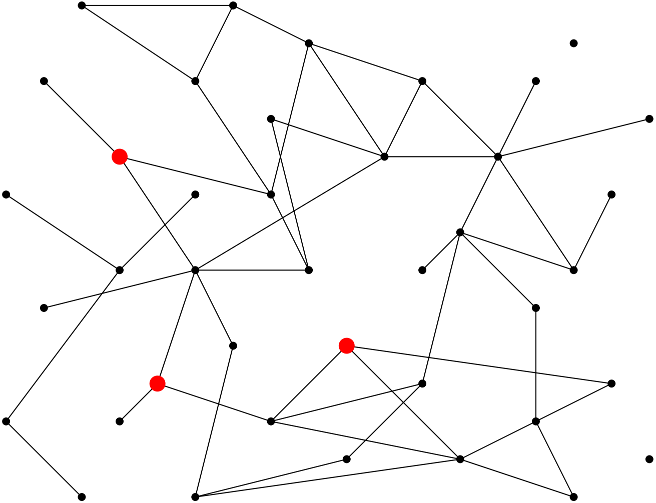

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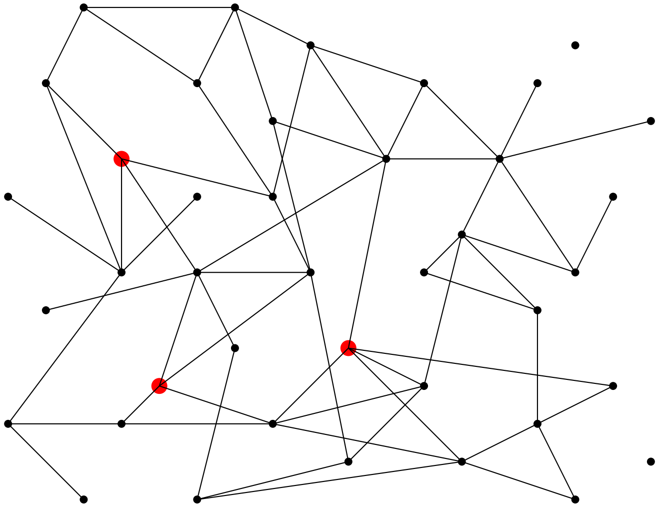

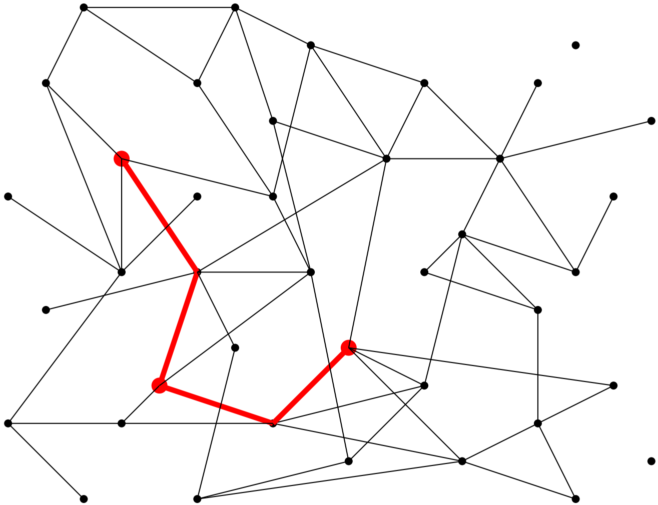

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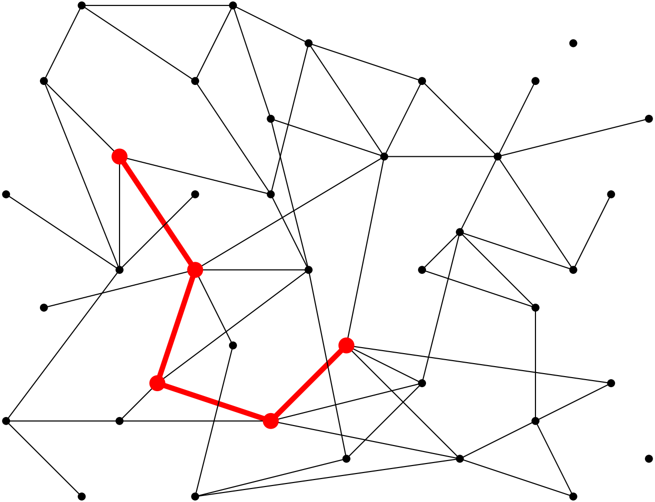

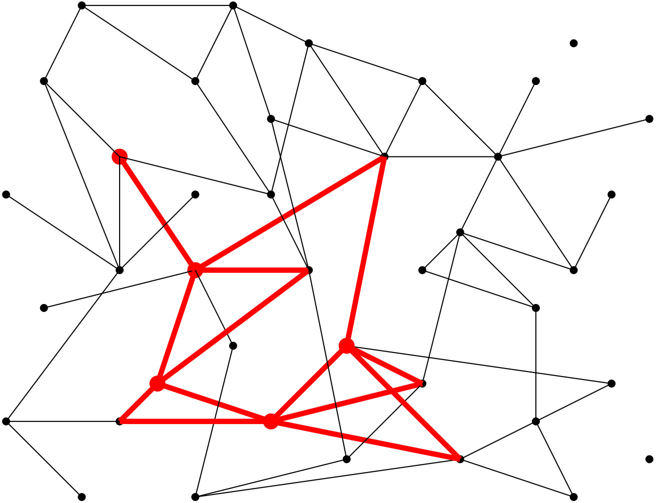

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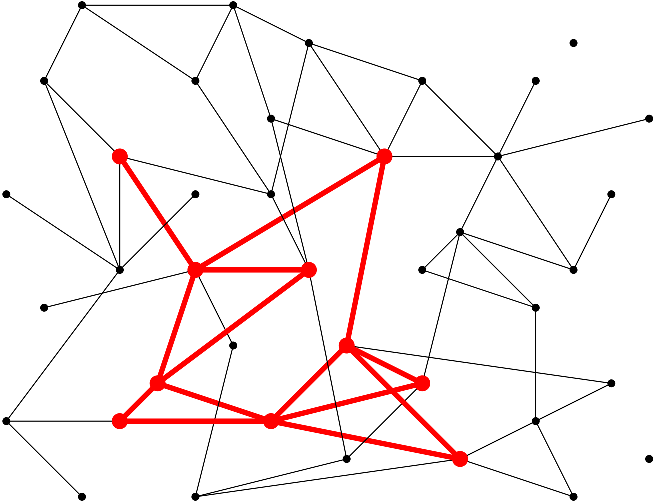

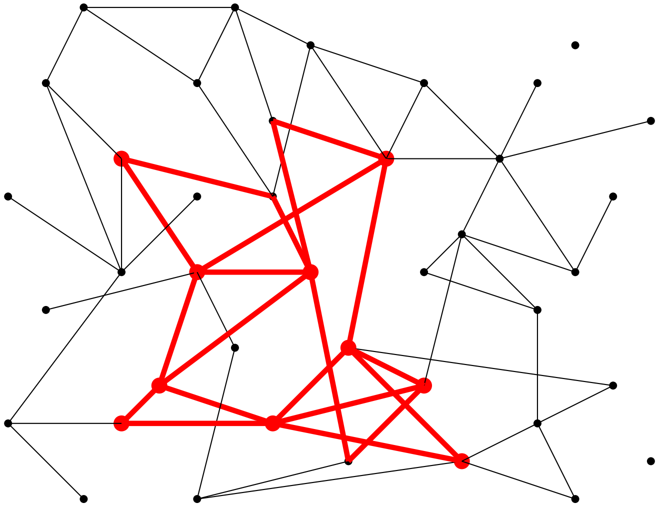

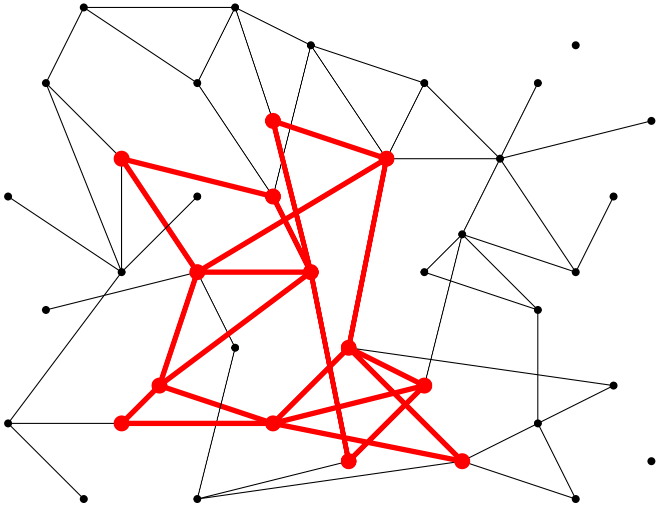

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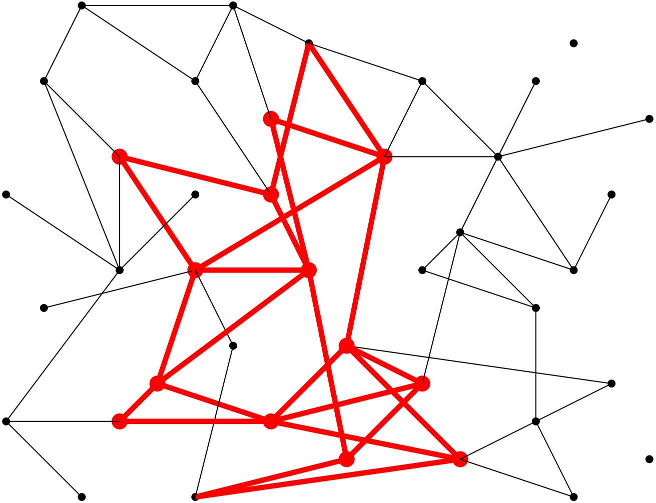

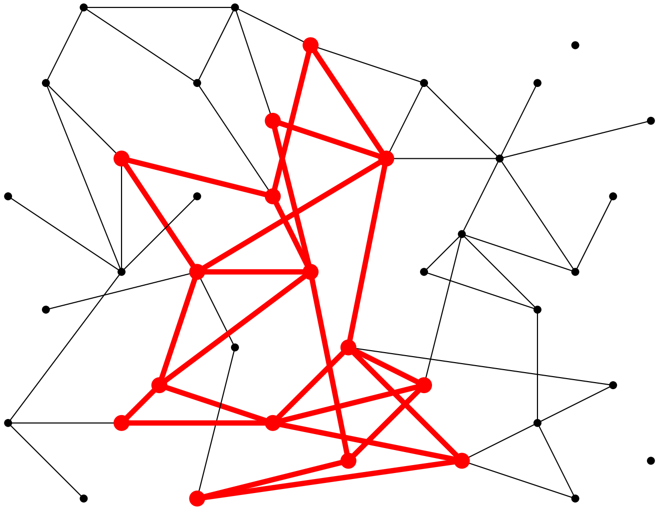

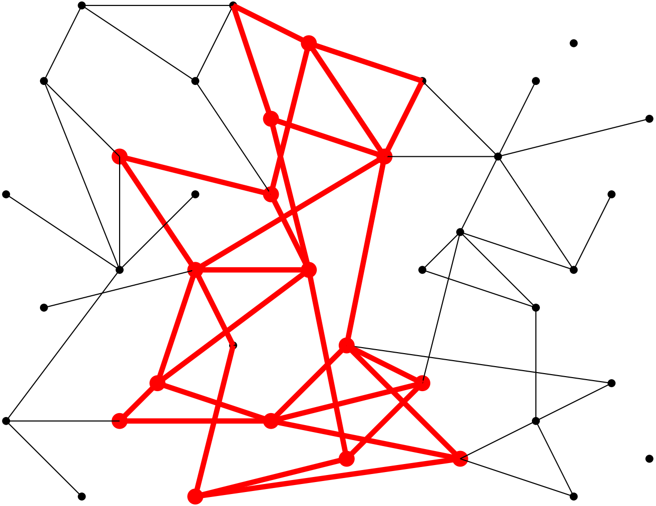

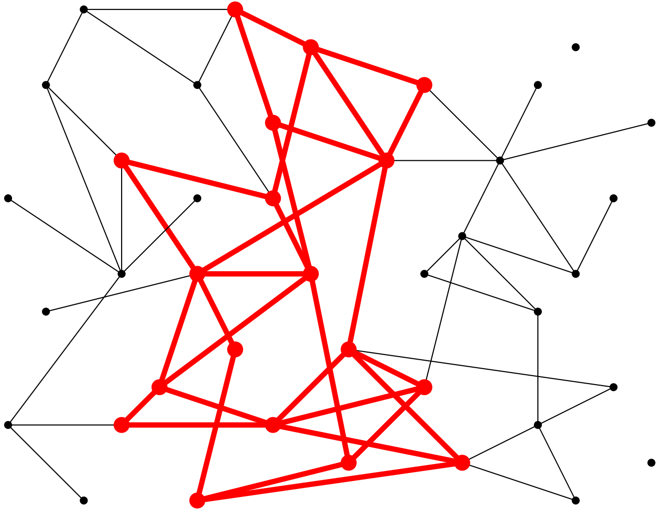

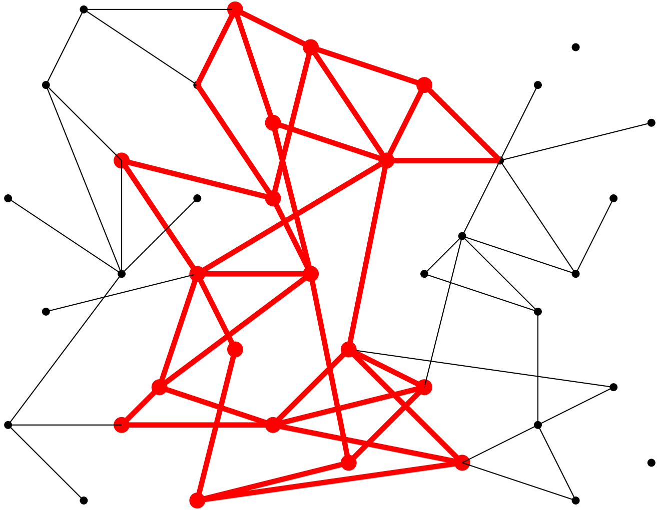

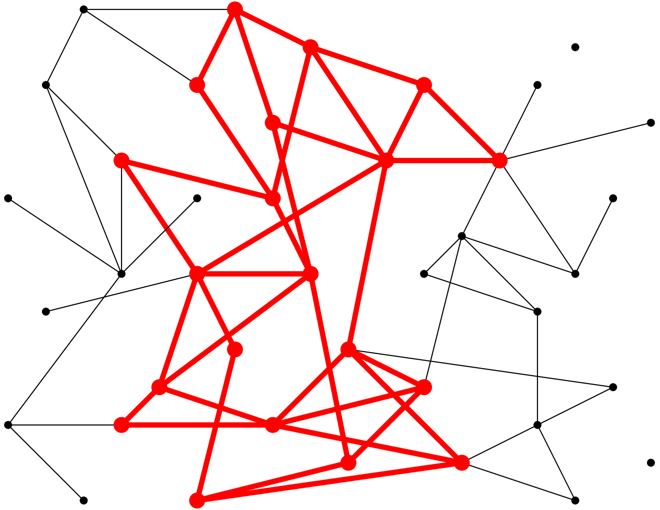

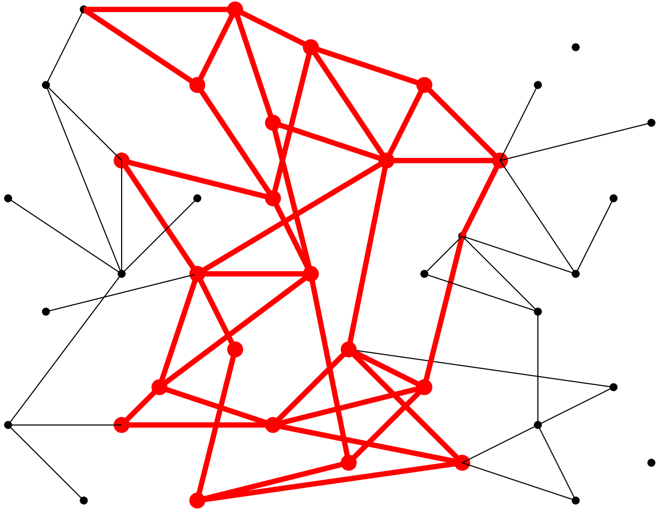

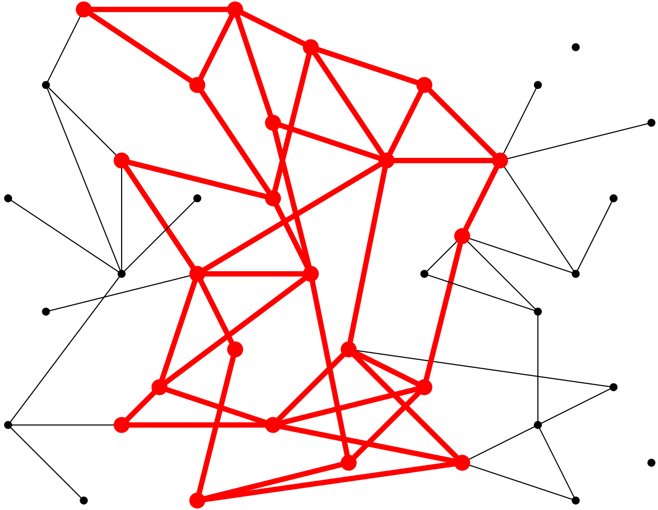

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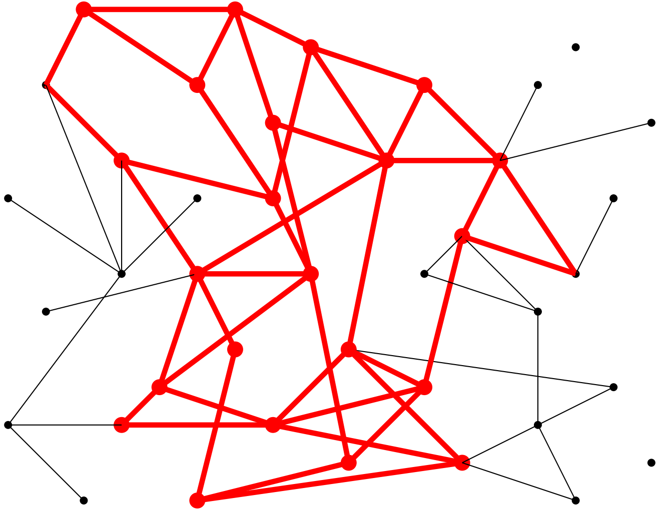

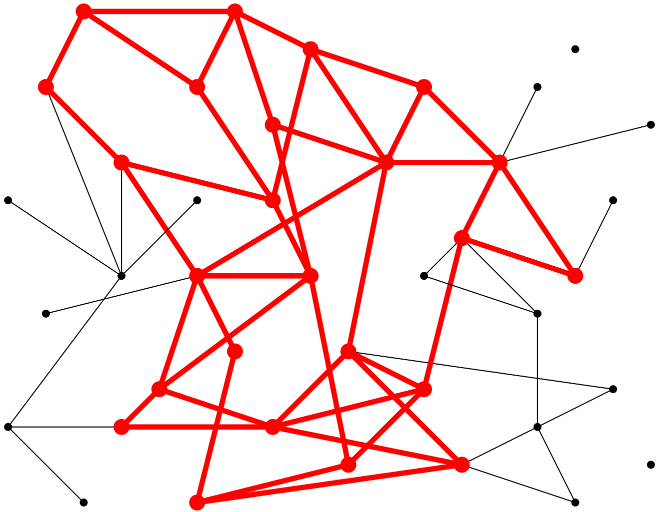

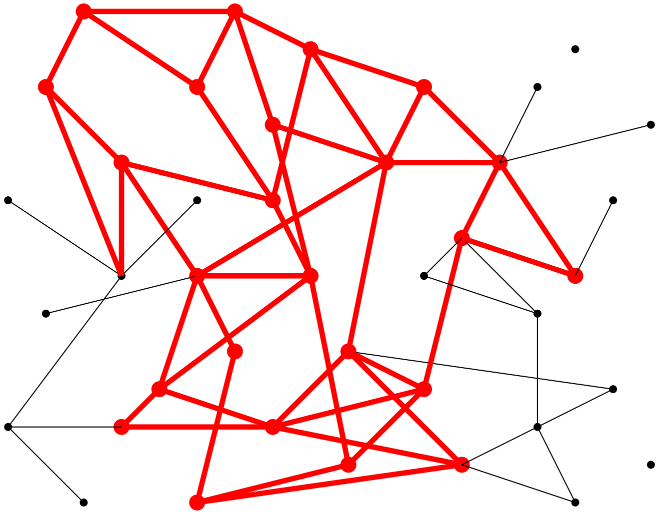

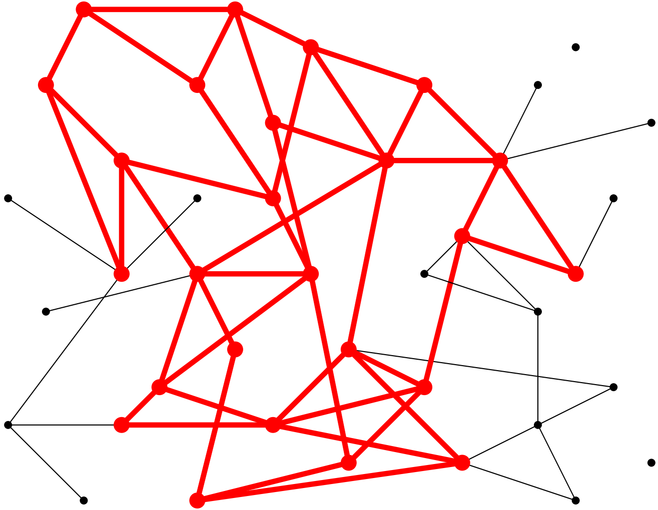

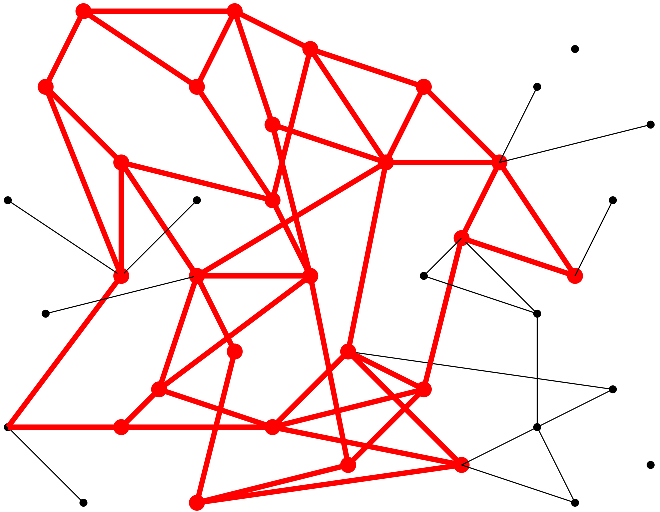

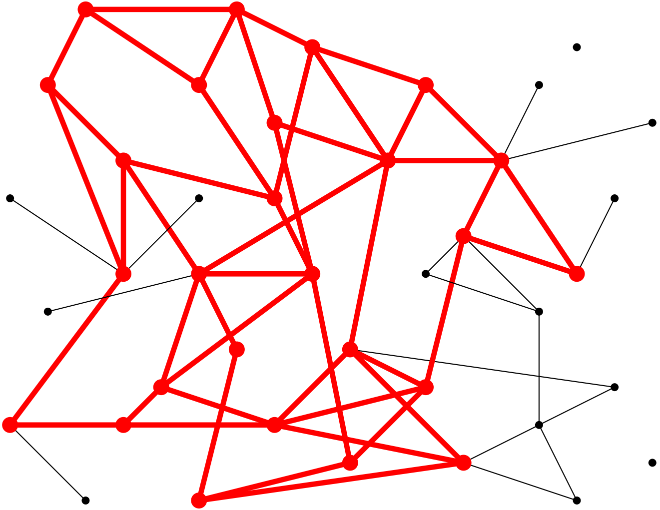

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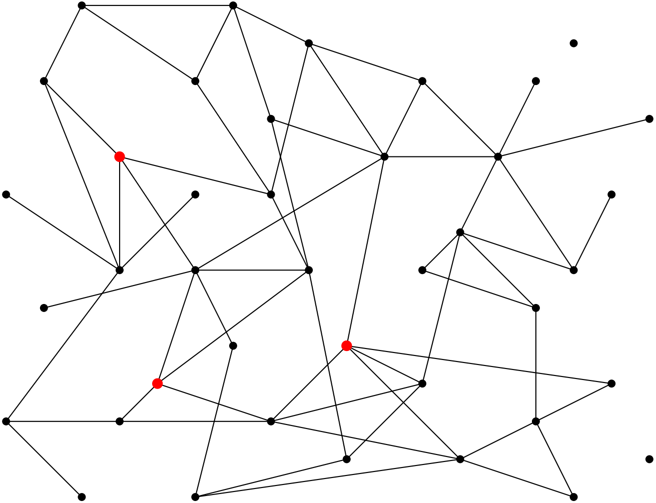

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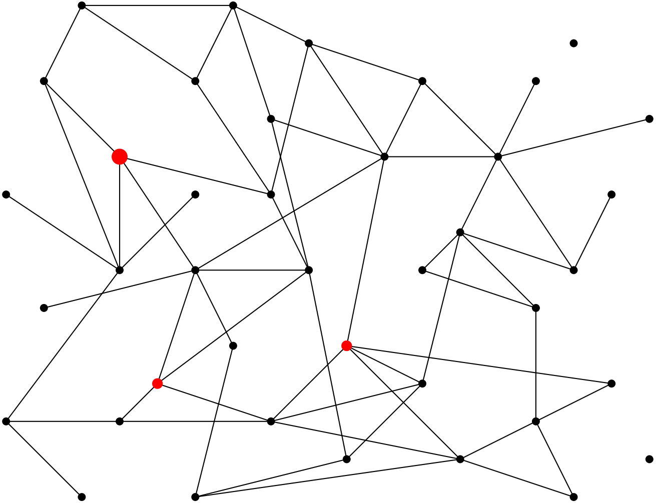

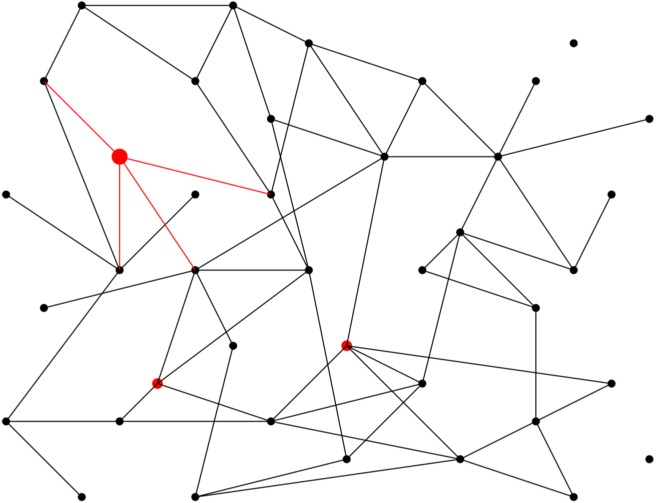

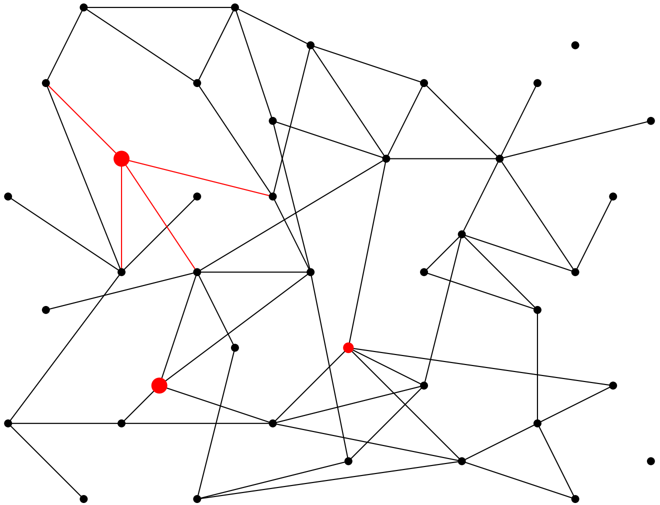

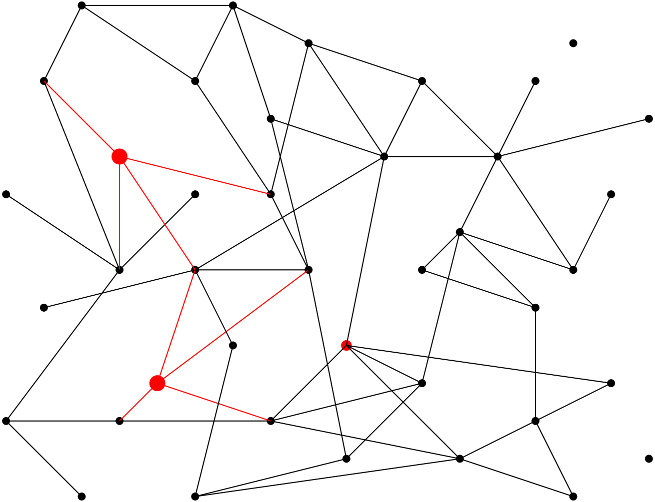

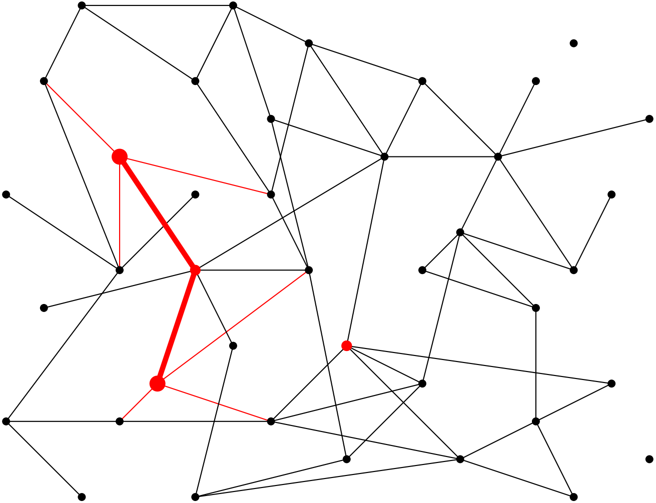

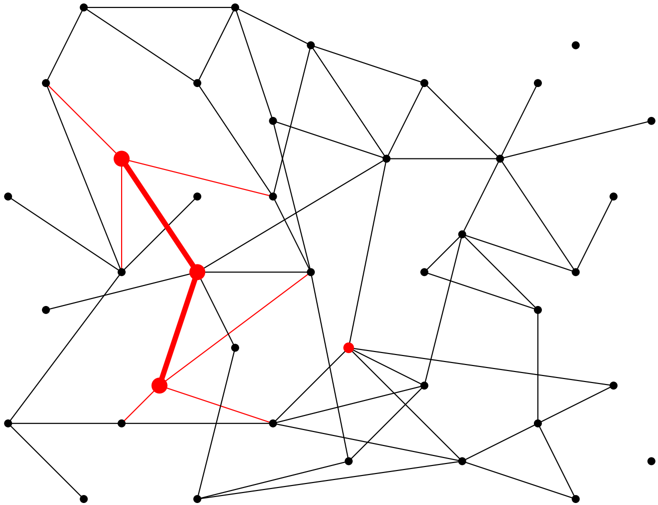

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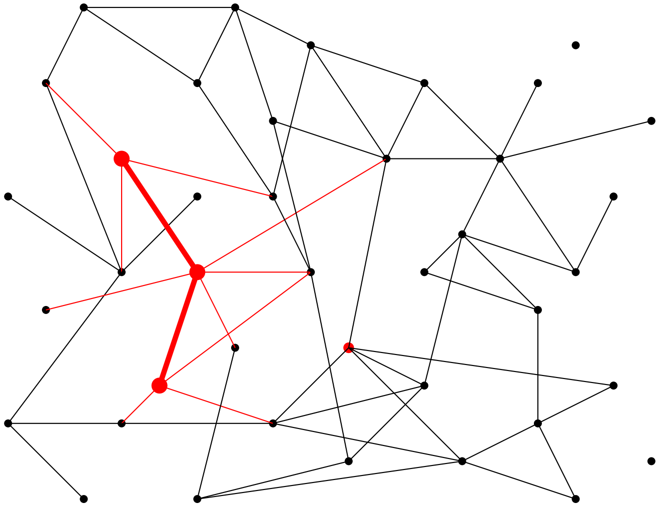

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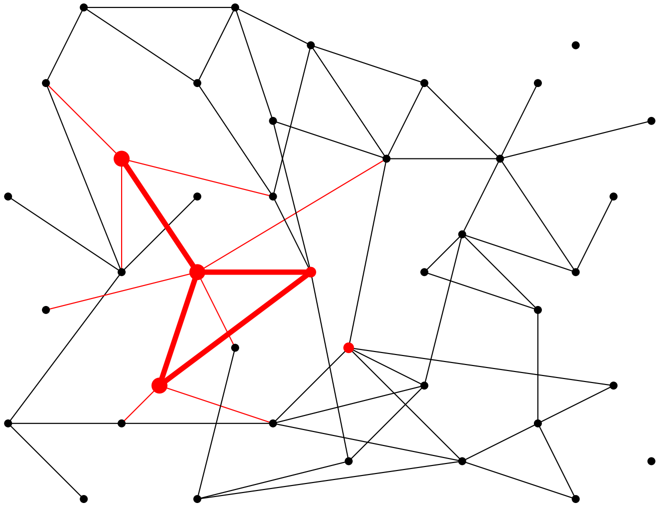

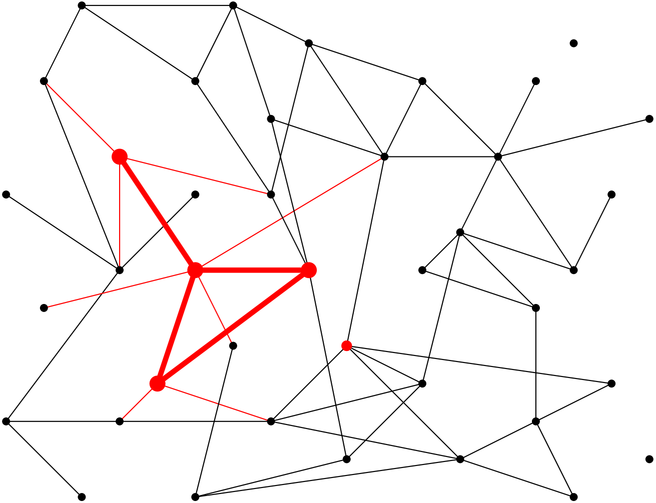

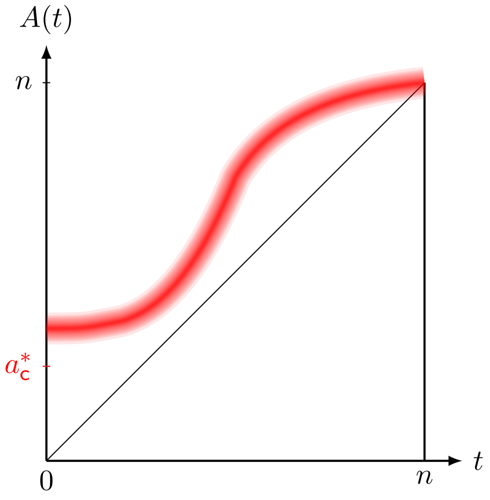


Figure: Bootstrap Percolation on $G_{n,p}$.

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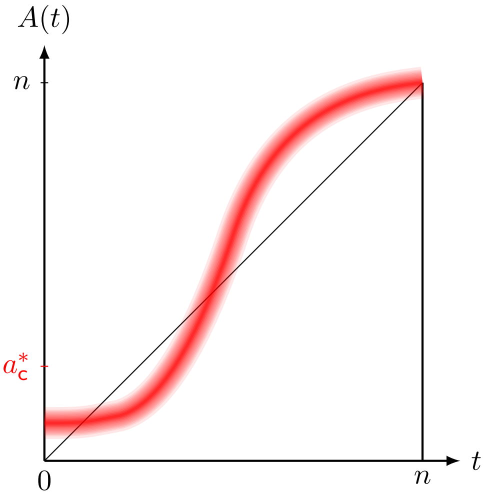


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*Suppose that* $k = 2$ *and* $n^{-1} \ll p \ll n^{-1/2}$. *There exist* $a_c^* \sim \frac{1}{2np^2}$ *and* $b_c = o(n)$ *such that,*

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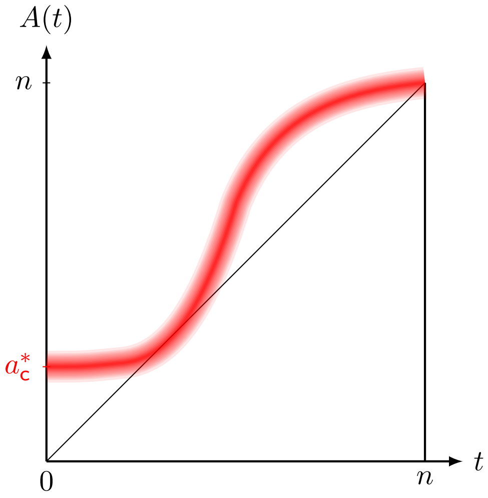


Figure: Around criticality.

## Theorem (Janson, Łuczak, Turova, V. (2012))

3. *If* $(a - a_c^*)/\sqrt{\frac{1}{2np^2}} \to y \in (-\infty, \infty)$, *then for every* $\varepsilon > 0$ *and every* $b^* \gg b_c$ *with* $b^* = o(n)$,

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$\Phi \in N(0,1)$.

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- S. Janson, T. Łuczak, T. Turova, T. Vallier Bootstrap percolation on $G_{n,p}$ *Ann. Appl. Prob* (2012)

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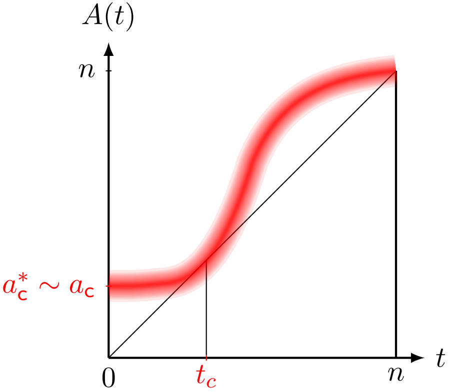


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