Compressibility of Behavioral Graphs

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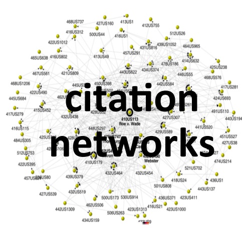
Behavioral graphs

- Web graphs
- Host graphs
- Social networks
- Collaboration networks
- Sensor networks
- Biological networks
- …



flickr™

social
networks

LiVEJOURNAL

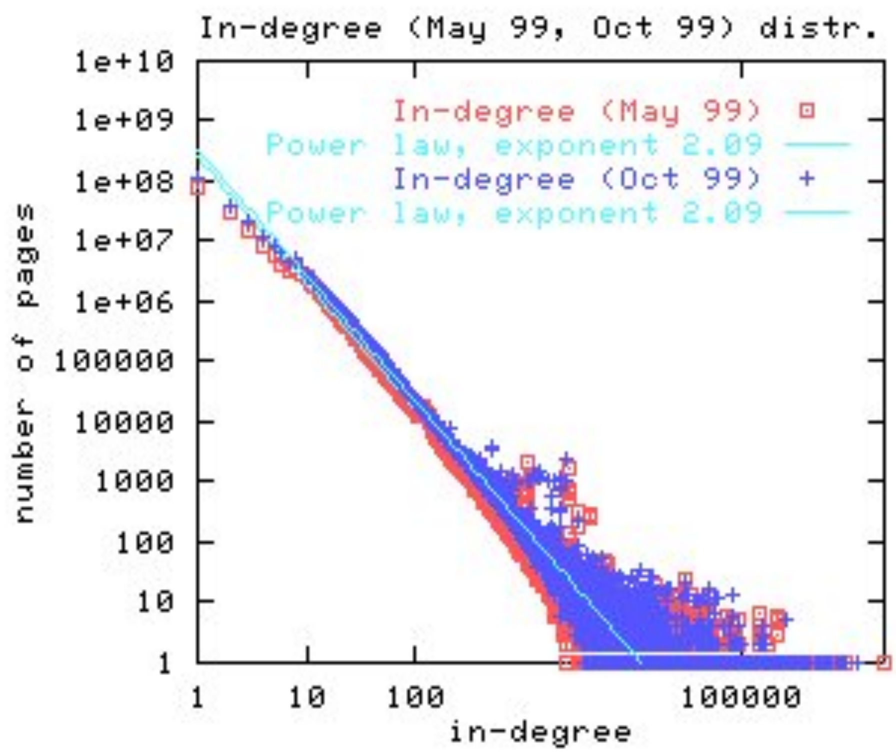
Research trends

- **Empirical analysis:** examining properties of real-world graphs
- **Modeling:** finding good models for behavioral graphs

There has been a tendency to lump together behavioral graphs arising from a variety of contexts

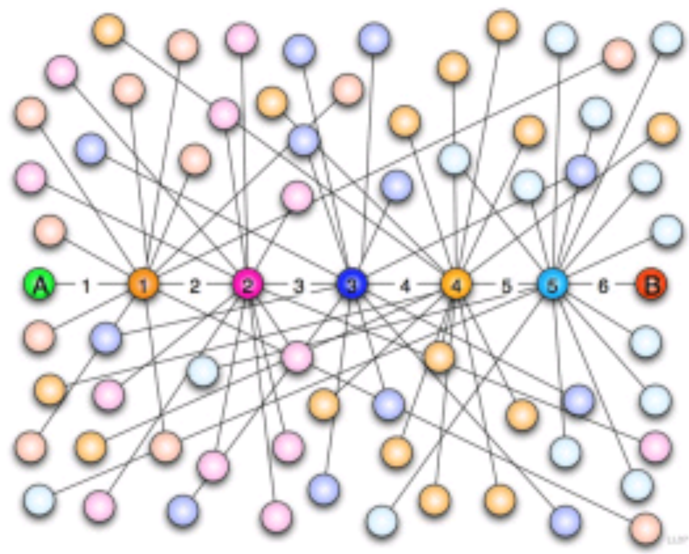
Properties of behavioral graphs

- Heavy-tail degree distributions, eg, power law $p(x) \propto x^{-\alpha}$



Other structural properties

- Clustering
  - High clustering coefficient
- Communities and dense subgraphs
  - Abundance; locally dense, globally sparse
- Connectivity
  - Exhibit a “bow-tie” structure; low diameter; small-world properties



A remarkable empirical fact

- **Snapshots of the web graph can be losslessly compressed using less than 3 bits per edge**

  Boldi, Vigna WWW 2004

- **Improved to ~2 bits using another data mining-inspired compression technique**

  Buehrer, Chellapilla WSDM 2008

- **Subsequent improvements**

  Boldi, Santini, Vigna WAW 2009

18.5 Mpages, 300 Mlinks from .uk									
R	Average reference chain			Bits/node			Bits/link		
	$W=1$	$W=3$	$W=7$	$W=1$	$W=3$	$W=7$	$W=1$	$W=3$	$W=7$
∞	171.45	198.68	195.98	44.22	38.28	35.81	2.75	2.38	2.22
3	1.04	1.41	1.70	62.31	52.37	48.30	3.87	3.25	3.00
1	0.36	0.55	0.64	81.24	62.96	55.69	5.05	3.91	3.46
Tranpose									
∞	18.50	25.34	26.61	36.23	33.48	31.88	2.25	2.08	1.98
3	0.69	1.01	1.23	37.68	35.09	33.81	2.34	2.18	2.10
1	0.27	0.43	0.51	39.83	36.97	35.69	2.47	2.30	2.22
118 Mpages, 1 Glinks from WebBase									
R	Average reference chain			Bits/node			Bits/link		
	$W=1$	$W=3$	$W=7$	$W=1$	$W=3$	$W=7$	$W=1$	$W=3$	$W=7$
∞	85.27	118.56	119.65	30.99	27.79	26.57	3.59	3.22	3.08
3	0.79	1.10	1.32	38.46	33.86	32.29	4.46	3.92	3.74
1	0.28	0.43	0.51	46.63	38.80	36.02	5.40	4.49	4.17
Tranpose									
∞	27.49	30.69	31.60	27.86	25.97	24.96	3.23	3.01	2.89
3	0.76	1.09	1.31	29.20	27.40	26.75	3.38	3.17	3.10
1	0.29	0.46	0.54	31.09	29.00	28.35	3.60	3.36	3.28

Why study compressibility?

- **Efficient storage**
  - **Serve adjacency queries in-memory – enables efficient algorithms**
  - **Archival purposes – store multiple snapshots efficiently**
- **Obtain new insights**
  - **Compression captures global network structure**
  - **Study the randomness in behavioral graphs**
  - **Validate existing graph models**
- **Algorithmic considerations**
  - **Possibility of working directly on compressed representations** Karande, Chellapilla, Andersen WSDM 2009

Adjacency list representation

- Each row corresponds to a node u in the graph
- Entries in a row are sorted integers, representing the neighborhood of u, ie, edges (u, v)

1: 1, 2, 4, 8, 16, 32, 64

2: 1, 4, 9, 16, 25, 36, 49, 64

3: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

4: 1, 4, 8, 16, 25, 36, 49, 64

- Can answer adjacency queries fast
- Expensive to store
  - Though, better than storing a list of edges

Neighborhood similarity

- **Similar neighborhoods:** Neighborhood of a web page can be expressed in terms of other web pages with similar neighborhoods
  - Rows in adjacency table have similar entries
  - Possible to choose a leader row

1:	1, 2, 4, 8, 16, 32, 64
2:	1, 4, 9, 16, 25, 36, 49, 64
3:	1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144
4:	1, 4, 8, 16, 25, 36, 49, 64

- **Locality:** Most edges are intra-host and hence local
  - Small integers can represent edge destination wrt source
- **Gap encoding:** Instead of storing destination of each edge, store the difference from the previous entry in the same row
  - Distribution of gap values: Optimal codes

The Boldi-Vigna scheme

Boldi-Vigna get down to an average of ~3 bits/ URL-URL edge, for an 118M node web graph

- **How does it work?**
- **Why does it work?**

Main ideas of Boldi-Vigna

Canonical ordering: **Sort URLs alphabetically, treating them as strings** Randall et al 2002

…

17: www.berkeley.edu/alchemy

18: www.berkeley.edu/biology

19: www.berkeley.edu/biology/plant

20: www.berkeley.edu/biology/plant/copyright

21: www.berkeley.edu/biology/plant/people

22: www.berkeley.edu/chemistry

…

This gives an identifier for each URL

Source and destination of edges are likely to get nearby IDs

- **Templated webpages**
- **Many edges are intra-host or intra-site**

Main ideas (contd)

- Due to templates, the adjacency list of a node is similar to one of the 7 preceding URLs in the alphabetic ordering
- Express adjacency list in terms of one of these
- Eg, consider these adjacency lists
  - 1: 1, 2, 4, 8, 16, 32, 64
  - 2: 1, 4, 9, 16, 25, 36, 49, 64
  - 3: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144
  - 4: 1, 4, 8, 16, 25, 36, 49, 64

Encode as (-2), remove 9, add 8

Gap encodings

- Given a sorted list of integers x, y, z, …, represent them by x, y-x, z-y, …
- Compress each integer using a code
  - γ code: x is represented by concatenation of unary representation of $\lfloor \lg x \rfloor$ (length of x in bits) followed by binary representation of $x - 2^{\lfloor \lg x \rfloor}$

    Number of bits = $1 + 2 \lfloor \lg x \rfloor$
  - δ code: …
  - Information theoretic bound: $1 + \lfloor \lg x \rfloor$ bits
  - ζ code: Works well for integers from a power law Boldi, Vigna DCC 2004

BV compression algorithm

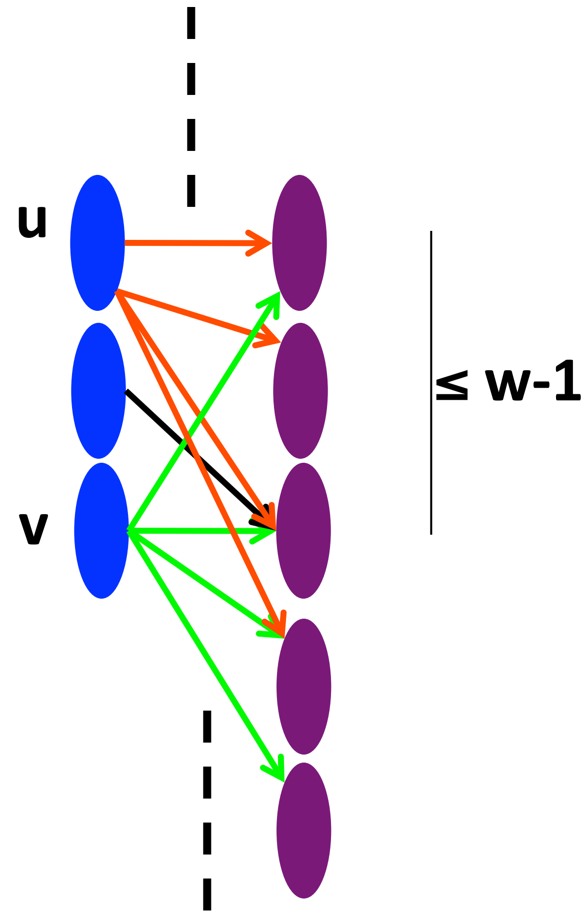
Each node has a unique ID from the canonical ordering

Let w = copying window parameter

To encode a node v

- Check if out-neighbors of v are similar to any of w-1 previous nodes in the ordering
- If yes, let u be the leader: use lg w bits to encode the gap from v to u + difference between out-neighbors of u and v
- If no, write lg w zeros and encode out-neighbors of v explicitly

Use gap encoding on top of this



Main advantages of BV

- Depends only on locality in a canonical ordering
  - Alphabetic ordering works well for web graph
- Adjacency queries can be answered very efficiently
  - To fetch out-neighbors, trace back the chain of leaders until a list whose encoding beings with lg w zeros is obtained (no-leader case)
  - This chain is typically short in practice (since similarity is mostly intra-host)
  - Can also explicitly limit the length of the chain during encoding
- Easy to implement and a one-pass algorithm

Practice vs Theory

Why does Boldi-Vigna compression work?

- **Similarity:** Many nodes have similar neighborhoods
- **Locality:** Most edges are local

Graph models and compression

- Are graphs generated by existing models compressible?
- Can we formulate a model with locality?

Social networks and compression

- Are social networks as compressible as the Web?

Preferential attachment model

Observation: Rich-get-richer Albert, Barabasi Science 1999

- Popular papers are cited more
- Popular people are befriended more

Each step has one new incoming node along with an edge

Probability this new node links to a pre-existing node is proportional to how popular is the latter, ie, its degree

$$\Pr[\text{new node links to node } i] = d_i / \sum d_j$$

Theorem. Degree distribution is a power law with exponent 3

Intuitive proof. $\partial d_i / \partial t = d_i / (2t)$

If node i was added at time t_i, then $d_i(t) = (t/t_i)^{0.5}$

$\Pr[d_i(t) > k] = \Pr[t_i < t/k^2] = 1/k^2$

Other "non-local" models

- **Copying model** Kumar et al FOCS 2000
  - **Observation:** People copy their friend's webpage when creating a new one or copy their friend's contacts when joining a social network
  - When a new node arrives, it copies edges from a pre-existing node with probability 1 - α
  - The degree distribution is a power-law with exponent

    $$(2 - \alpha)/(1 - \alpha)$$

  - Can explain communities: The number of dense bipartite cliques in this model is large
- **Forest-fire model** Leskovec, Kleinberg, Faloutsos KDD 2005
  - An iterated version of the copying model
  - In addition to the above, leads to densification and shrinking diameters

Incompressibility Chierichetti et al FOCS 2009

Theorem. The following generative models all require $\Omega(\log n)$ bits per edge on average, even if the node labels are removed

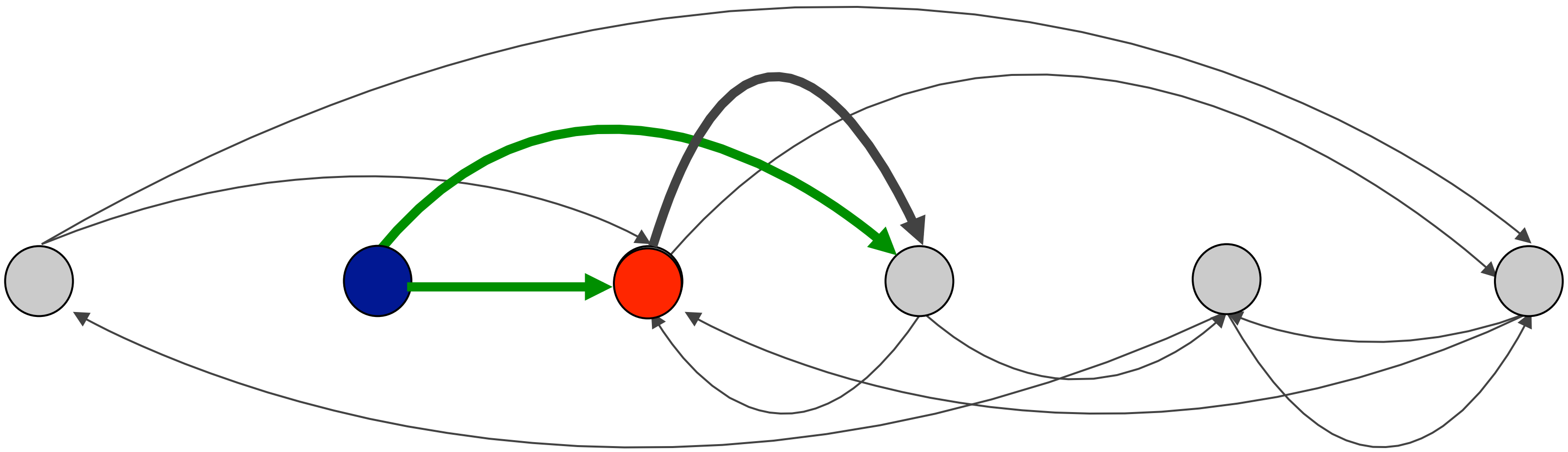
- the preferential attachment model
- the copying model
- the evolutionary ACL model Aiello, Chung, Lu FOCS 2001
- Kronecker multiplication model Leskovec et al PKDD 2005
- Model for navigability in social networks Kleinberg Nature 2000

- We remove labels since BV compresses unlabeled Web graphs to O(1) bits per edge
- Min-entropy argument: Find a subset of graphs
  - not too large: to avoid graphs that are "easy"
  - not too small: should still contain interesting graphs about which we can show incompressibility

A new graph model Chierichetti et al FOCS 2009

- Begin with a seed graph of nodes with out-degree k, arranged in a cycle
- Additional nodes arrive in sequence
- An arriving node is inserted before a random node in the cycle (*leader*)
  - It links to k-1 out-neighbors of its leader
  - It links to the leader

An example, k=2



Locality in the new model

- If a web designer wants to add a new web page to her web site
  - likely to take some existing web page on her website
  - modify it as needed (perturbing the set of its outlinks) to obtain the new page
  - adding a reference to the old web page
  - and publish the new web page on her website
- Since web pages are sorted by URL in our ordering, the old and the new page will be close!

Basic properties of the model

- **Rich get richer:** in the model, in-degrees converge to a power law with exponent **-2-1/(k-1)**
- High clustering coefficient
- Polynomially many bipartite cliques
- Logarithmic undirected diameter

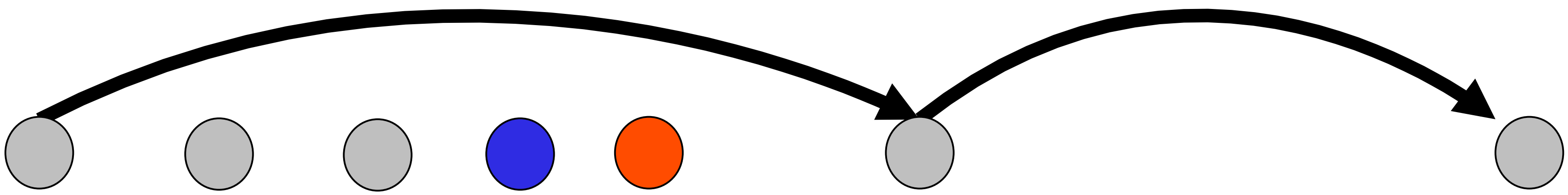
- Compressible to O(1) bits per edge
- In fact, BV algorithm achieves O(1) bits per edge

Compressibility

- **Theorem.** The number of bits required by BV algorithm is $\sum_{l=1..\infty} Y_l (\log l)$, where Y_l is the number of edges of length l
- **Theorem.** In the model, edge lengths converge to a power law with exponent $-1-1/k$

- **Corollary.** The new model produces graphs compressible to O(1) bits per edge

Long gets longer



- Recall the process: pick a leader node uniform at random and place new node to its immediate left
- The probability to become longer is proportional to the number of nodes "below" the edge, ie, its length
- Making this precise requires pinning down subtle combinatorial properties of the model

Are social networks compressible?

- How does BV perform on social networks?
- Can we take use special properties, eg, social networks are highly reciprocal, despite being directed
  - If A is a friend of B, then it is likely B is also A's friend
- How to exploit reciprocity in compression?
  - Can avoid storing reciprocal edges twice
  - Just the reciprocity “bit” is sufficient
  - Modify BV to get a new scheme

Canonical orderings

- **BV compressions depend on a canonical ordering of nodes**
  - **This canonical ordering should exploit neighborhood similarity and edge locality**
- **How do we get a good canonical ordering?**
  - **Unlike the web page case, it is unclear if social networks have a natural canonical ordering**
- **Caveat: BV is only one genre of compression scheme**
  - **Lack of good canonical ordering does not mean graph is incompressible**

Some natural canonical orderings

- Random order
- Natural order
  - Time of joining in a social network
  - Lexicographic order of URLs
  - Crawl order
- Graph traversal orders
  - BFS and DFS
- Use attributes of the nodes
  - Eg, Geographic location: order by zip codes
  - May produce a bucket order
- Ties can be broken using more than one order

Performance of simple orderings

Graph	#nodes	#edges	%reciprocal edges
Flickr	25.1M	69.7M	64.4
UK host graph	0.58M	12.8M	18.6
IndoChina	7.4M	194.1M	20.9

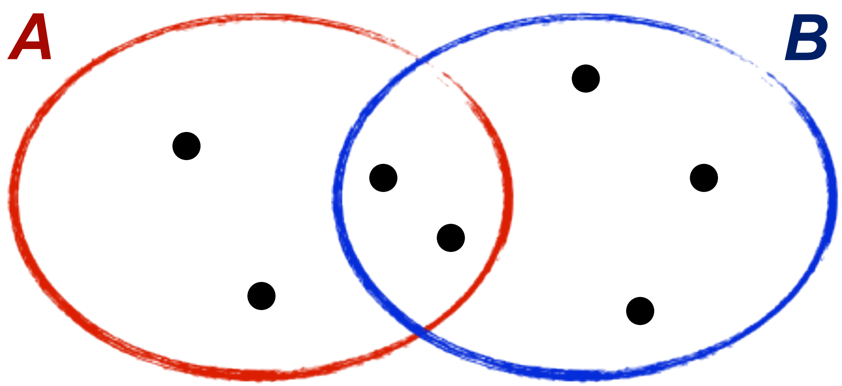
Graph	Natural	Random	DFS
Flickr	21.8	23.9	22.9
UK host	10.8	15.5	14.6
IndoChina	2.02	21.44	-

Detour: Shingles

- **Jaccard coefficient:** Measures similarity between sets A and B

  $J(A, B) = |A \cap B| / |A \cup B|$

- 1 – J(A, B) is a metric

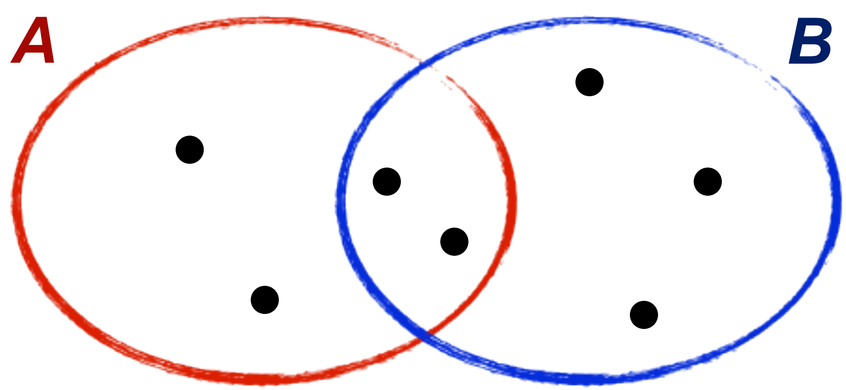


MinHash fingerprinting Broder

- Can we construct a hash function h such that

$$Pr[h(A) = h(B)] = |A \cap B| \ / \ |A \cup B| = J(A, B)$$

- Given a universe U, pick a permutation π on U uniformly at random
- Hash each subset $S \subseteq U$ to the minimum value it contains according to π



Shingle ordering heuristic

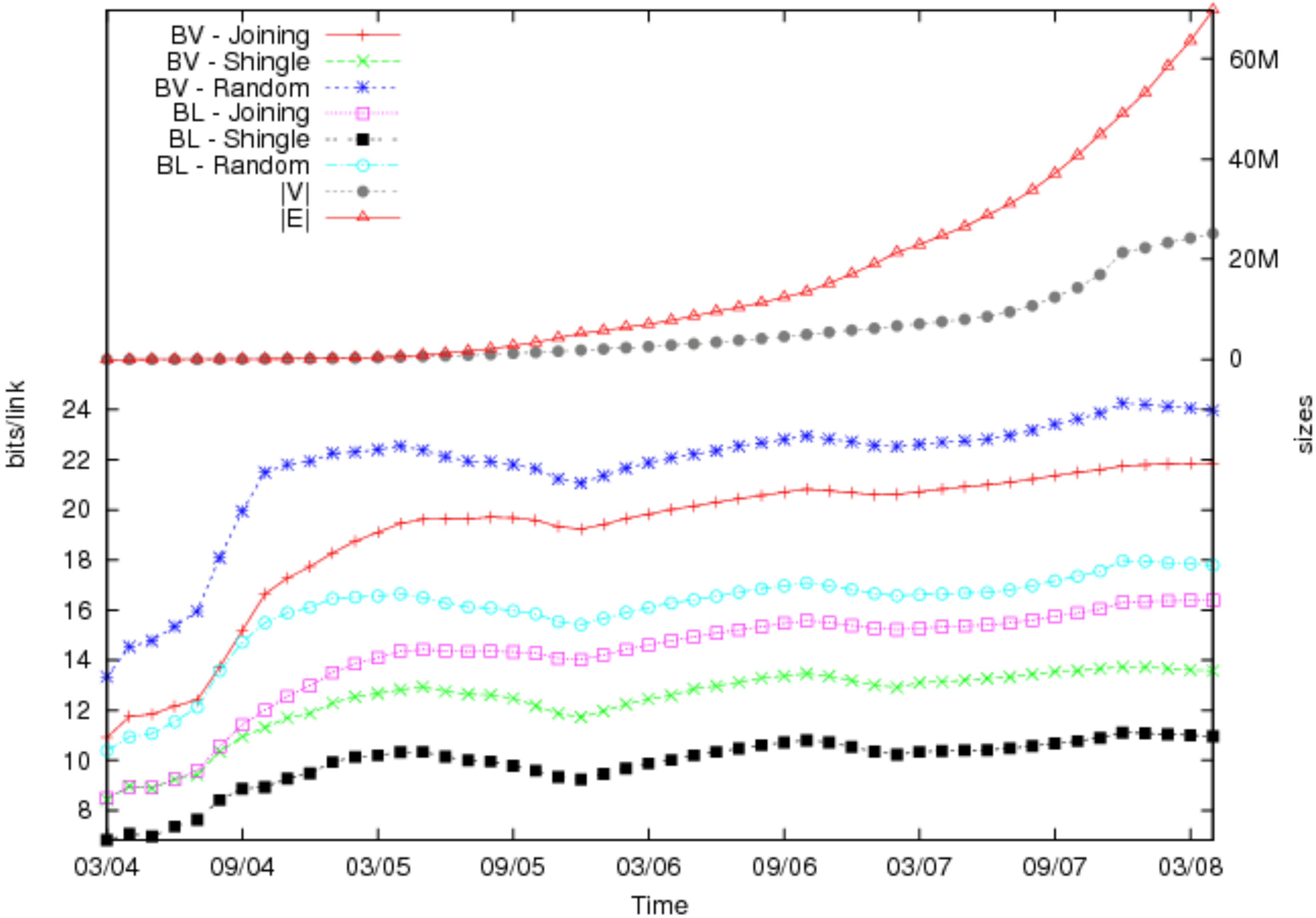
- Chierichetti et al KDD 2009
- Obtain a canonical ordering by bringing nodes with similar neighborhoods close together
- Fingerprint neighborhood of each node
  - Order the nodes according to the fingerprint
  - If fingerprint can capture neighborhood similarity and edge locality, then it can produce good compression via BV
- Double shingle order: break ties within shingle order using a second shingle

Performance of shingle ordering

Graph	Natural	Shingle	Double shingle
Flickr	21.8	13.5	13.5
UK host	10.8	8.2	8.1
IndoChina	2.02	2.7	2.7

Geography does not seem to help for Flickr graph

Flickr: Compressibility over time

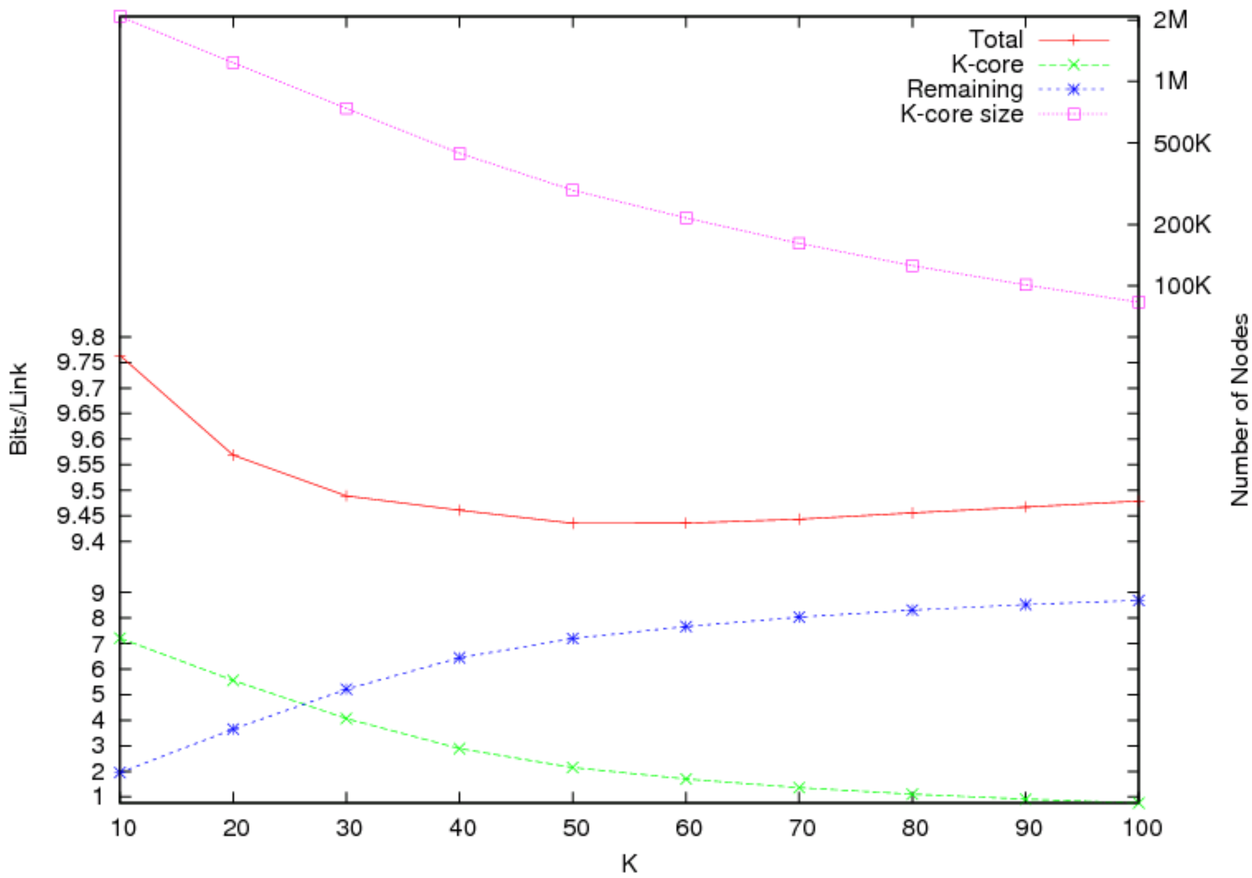


A property of shingle ordering

Theorem. Using shingle ordering, a constant fraction of edges will be "copied" in graphs generated by preferential attachment/copying models

- **Preferential attachment model:** Rich get richer – a new node links to an existing node with probability proportional to its degree
- Shows that shingle ordering helps BV-style compressions in stylized graph models

Who is the culprit



Low degree nodes are responsible for incompressibility

Compression-friendly orderings

Chierichetti et al KDD 2009

In BV, canonical order is all that matters

Problem. Given a graph, find the canonical ordering that will produce the best compression in BV

- The ordering should capture locality and similarity
- The ordering must help BV-style compressions

- We propose a formulation of this problem
- Recent developments
  - Gray-code ordering Boldi, Santini, Vigna IM 2010
  - Multi-scale ordering Safro, Temkin JDA 2010
  - Layered Label Propagation Boldi, Rosa, Santini, Vigna WWW 2011

MLogGapA formulation

MLogGapA. For an ordering π, let $f_\pi(u)$ = cost of compressing the out-neighbors of u under π

If $u_1, \ldots, u_k$ are out-neighbors ordered wrt π, $u_0 = u$

$$f_\pi(u) = \sum_{i=1..k} \lg |\pi(u_i) - \pi(u_{i-1})|$$

Find an ordering π of nodes to minimize

$$\sum_u f_\pi(u)$$

- Minimize encoding gaps of neighbors of a node

Theorem. MLinGapA is NP-hard

Conjecture. MLogGapA is NP-hard

Summary

- Social networks appear to be not very compressible, but the Web graph is
  - Both exhibit “local” power laws
  - Host graphs are equally challenging
- BV compression
  - Optimal orderings
  - Combinatorial formulations and heuristics
- Generative models
  - Lower bounds for prior models
  - New compressible model

Future directions

- Can we compress social networks better?
- Is there a lower bound on incompressibility?
  - Our analysis applies only to BV-style compressions
- Algorithmic questions
  - Hardness of MLogGapA
  - Good approximation algorithms for good orderings
  - Algorithms that work on compressed graphs
- Modeling questions
  - More nuanced, tractable models for compressibility

Thank you!

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