Math 104: Solutions to sample final problems

1. For the first series

$$\beta = \limsup |a_n|^{1/n} = \limsup |n|^{3/n} = 1$$

so the radius of convergence is $R = {}^1/_\beta = 1$. At $n = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

which converges (and can be verified using the integral test). At $n = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

which converges by the alternating series test. Hence the exact interval of convergence is $[-1, 1]$. For the second series, only every third term is non-zero, and thus

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_{3n}|^{1/3n} = \lim_{n\to\infty} \left| \frac{1}{2n} \right|^{1/3n} = 1$$

so the radius of convergence is 1. At $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

which diverges. At $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

which converges by the alternating series test. Hence the exact interval of convergence is $(-1, 1]$. For the third series

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_{2n!}|^{1/2n!} = \lim_{n\to\infty} 1 = 1$$

and hence the radius of convergence is 1. At $x = 1$

$$\sum_{n=0}^{\infty} x^{2n!} = \sum_{n=0}^{\infty} 1$$

which diverges. Since the series is even, it diverges at $x = -1$ also. Hence the exact interval of convergence is $(-1, 1)$.

2. (a) Suppose that $s_n \to s$. Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|s_n - s| < \epsilon.$$

To show that $s_n^2 \to s^2$, consider any $\epsilon > 0$. First, note that

$$|s_n^2 - s^2| = |s_n - s| \cdot |s_n + s|.$$

There exists an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that $|s_n - s| < 1$, and hence that

$$|s_n + s| \leq |s_n| + |s| < (|s| + 1) + |s| = 2|s| + 1.$$

Similarly, there exists an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that

$$|s_n - s| < \frac{\epsilon}{2|s| + 1}.$$

Hence, if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$\begin{aligned} |s_n^2 - s^2| &\leq |s_n - s| \cdot |s_n + s| \\ &< \frac{\epsilon}{2|s| + 1}(2|s| + 1) = \epsilon. \end{aligned}$$

(b) A function f is continuous at a point x if for all sequences (x_n) such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$. By part (a), for all $x \in \mathbb{R}$, if $x_n \to x$, then $x_n^2 \to x^2$. Thus f is continuous for all $x \in \mathbb{R}$.

Alternatively, to use the ϵ-δ property, choose $\epsilon > 0$, and fix $x_0 \in \mathbb{R}$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|.$$

If $|x - x_0| < 1$, then $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$. Pick $\delta = \min\{\frac{\epsilon}{2|x_0|+1}, 1\}$. Then $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \frac{\epsilon}{2|x_0| + 1}(2|x_0| + 1) = \epsilon$$

and hence f is continuous at x_0. Since x_0 is arbitrary, f is continuous on $\mathbb{R}$.

3. By the mean value theorem, there exists a $c \in (r, s)$ such that

$$f'(c) = \frac{f(s) - f(r)}{s - r} = 0$$

and there exists and $d \in (s, t)$ such that

$$f'(d) = \frac{f(t) - f(s)}{t - s} = 0.$$

By construction, $d > c$. Hence there exists an $x \in (c, d) \subseteq (0, 1)$ such that

$$f''(x) = \frac{f'(d) - f'(c)}{d - c} = 0.$$

4. (a) If $\sum_{n=0}^{\infty} a_n$ converges to a limit a, then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $k > N$ implies

$$\left| a - \sum_{n=0}^{k} a_n \right| < \epsilon.$$

Now consider any $\epsilon > 0$ and define N as above. If $k > N/2$, then

$$\left| a - \sum_{n=0}^{k} b_n \right| = \left| a - \sum_{n=0}^{2k+1} a_n \right| < \epsilon.$$

Hence $\sum_{n=0}^{\infty} b_n$ converges.

(b) Suppose $a_n = (-1)^n$. Then

$$\sum_{n=0}^{N} a_n = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd;} \end{cases}$$

this series diverges. However $b_n = 0$ for all n, and thus $\sum_{n=0}^{\infty} b_n$ converges.

5. If $f(a)f(b) < 0$, then it follows that either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. In either case, zero lies between $f(a)$ and $f(b)$, and thus the intermediate value theorem can be applied to show that there exists an $x \in (a, b)$ such that $f(x) = 0$.

6. Let f be a real-valued function defined on an interval $[0, b]$ as

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Consider a partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$. First note that

$$m(f, [t_{k-1}, t_k]) = 0, \qquad M(f, [t_{k-1}, t_k]) = t_k$$

for all $k = 1, \ldots, n$. The first expression follows because any interval must contain an irrational number, whereas the second expression follows because the interval must take rational values arbitrarily close to t_k. Hence

$$U(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = \sum_{k=1}^{n} (t_k - t_{k-1}) t_k.$$

For a general partition this cannot be calculated explicitly, but it will always be strictly positive, since each term in the sum is strictly positive. The lower Darboux sum is given by

$$L(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = 0.$$

To show that f is not integrable on $[0, b]$, consider any partition P. Then

$$\begin{aligned}
U(f, P) &= \sum_{k=1}^{n} (t_k - t_{k-1}) t_k \\
&\geq \sum_{k=1}^{n} (t_k - t_{k-1}) \frac{(t_k + t_{k-1})}{2} \\
&= \frac{1}{2} \sum_{k=1}^{n} (t_k^2 - t_{k-1}^2) \\
&= \frac{t_n^2 - t_0^2}{2} = \frac{b^2}{2}.
\end{aligned}$$

Thus for any partition,

$$U(f, P) - L(f, P) \geq \frac{b^2}{2}$$

and thus f is not integrable on $[0, b]$.

7. Suppose that $\sum_{n=1}^{\infty} f(n)$ converges, and consider the integral

$$\int_1^b f(x)\, dx.$$

Let k be the largest integer such that $k < b$. Hence $b - k \leq 1$. Consider the partition $P = \{1 = t_0 < t_1 < \ldots < t_{k-1} < t_k = b\}$, where $t_i = i + 1$ for $i = 0, \ldots, k-1$. Then

$$\begin{aligned}
U(f, P) &= \sum_{j=1}^{k} M(f, [t_{j-1}, t_j])(t_j - t_{j-1}) \\
&= \sum_{j=1}^{k} f(t_{j-1})(t_j - t_{j-1}) \\
&= f(k)(b - k) + \sum_{j=1}^{k-1} f(j) \\
&\leq f(k) + \sum_{j=1}^{k-1} f(j) \\
&= \sum_{j=1}^{k} f(j)
\end{aligned}$$

so

$$\int_1^b f \leq U(f, P) \leq \sum_{j=1}^{k} f(j) \leq \sum_{j=1}^{\infty} f(j).$$

Since the integral is an increasing function and is bounded above, it follows that it must converge.

Now suppose that $\int_1^x f$ converges, and consider $\sum_{n=1}^{N} f(n)$. Consider the partition $P = \{1 = t_0 < t_1 < \ldots < t_N - 1 = N\}$ where $t_i = i + 1$ for all $i = 0, \ldots, N-1$. Then

$$\begin{aligned} L(f,P) &= \sum_{j=1}^{N-1} m(f,[t_{j-1},t_j])(t_j - t_{j-1}) \\ &= \sum_{j=1}^{N-1} m(f,[j,j+1]) \\ &= \sum_{j=1}^{N-1} f(j+1) \\ &= \sum_{j=2}^{N} f(j) \end{aligned}$$

and hence

$$\sum_{j=1}^{N} f(j) = f(1) + L(f,P) \leq f(1) + \int_1^N f \leq f(1) + \int_1^\infty f.$$

Since $\sum_{j=1}^{N} f(j)$ is increasing and bounded above, then it converges.

8. (a) From the definition, it is clear that $d(x,y) = 0$ if and only if $x = y$, and that $d(x,y) = d(y,x)$. To prove the triangle inequality, consider any $x, y, z \in \mathbb{R}$. If $x = z$, then $d(x,z) = 0$, and since $d(x,y) + d(y,z) \geq 0$, the triangle inequality is satisfied. If $x \neq z$, then $d(x,z) = 1$. Either $y \neq z$ or $y \neq x$, and hence $d(x,y) + d(y,z) \geq 1$, so the triangle inequality is satisfied. Hence d is a metric.

   (b) The neighborhood of radius $1/2$ at 0 is

   $$\{x \in \mathbb{R} : d(0,x) < 1/2\} = \{0\}.$$

   It only contains zero, since all other points are a distance of 1 away.

   (c) Consider an arbitrary set $S \subseteq \mathbb{R}$. To show that S is open, consider any $x \in S$. Then, by the same argument as in (b), $N_{1/2}(x) = \{x\} \subseteq S$ so x is an interior point. Since all points are interior, it follows that S is open.

   Consider an open cover $\mathcal{S}$ of S. Suppose S has finitely many points, so that $S = \{s_1, s_2, \ldots, s_n\}$. Then there exist sets $S_1, \ldots, S_n \in \mathcal{S}$ such that $s_k \in S_k$ for $k = 1, \ldots, n$, and it follows that $\{S_k : k = 1, \ldots n\}$ is a finite subcover. Hence S is compact.

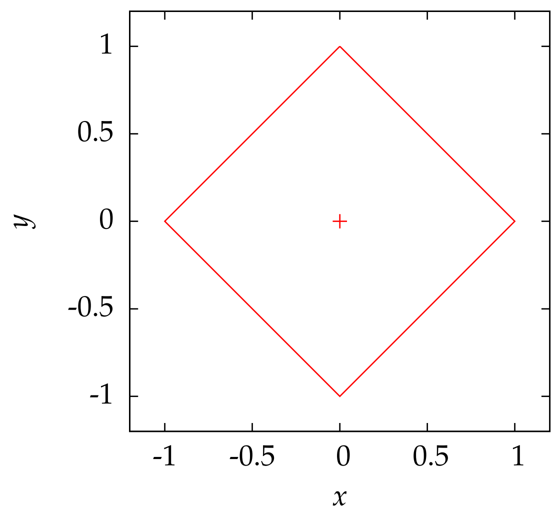

Figure 1: A plot of the neighborhood of radius 1 at $(0,0)$, considered in Problem 9(b).

Suppose S has infinitely many points. Consider the cover $\mathcal{S} = \{\{x\} : x \in S\}$. By above, the sets $\{x\}$ are open. Consider a subcover $\mathcal{T} \subseteq \mathcal{S}$. For any $x \in S$, $\{x\}$ must be $\mathcal{T}$, since x is not an element of any other set in $\mathcal{S}$. Thus $\mathcal{T} = \mathcal{S}$. Hence $\mathcal{S}$ has no finite subcover and $\mathcal{S}$ is not compact.

9. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be in $\mathbb{R}^2$. Consider the function

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

(a) Consider the three properties of a metric:

M1. Note that

$$d(\mathbf{x}, \mathbf{x}) = |x_1 - x_1| + |x_2 - x_2| = 0$$

and if $d(\mathbf{x}, \mathbf{y}) = 0$, then

$$0 = |x_1 - y_1| + |x_2 - y_2|$$

from which it follows that $x_1 = y_1$ and $x_2 = y_2$, so $\mathbf{x} = \mathbf{y}$.

M2. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(\mathbf{y}, \mathbf{x}),$$

and thus d is symmetric.

M3. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\leq |x_1 - z_1| + |x_2 - z_2| \\ &= d(\mathbf{x}, \mathbf{z}), \end{aligned}$$

where the usual triangle inequality on $\mathbb{R}$ has been applied twice. Hence d satisfies the triangle inequality.

(b) The neighborhood of radius 1 at $(0,0)$ is given by

$$N_1((0,0)) = \{\mathbf{x} \in \mathbb{R}^2 \,:\, d(\mathbf{x},(0,0)) < 1\} = \{\mathbf{x} \in \mathbb{R}^2 \,:\, |x_1| + |x_2| < 1\}$$

Consider the quadrant where $x_1 \geq 0$ and $x_2 \geq 0$. Then $|x_1| + |x_2| = x_1 + x_2$, and thus the boundary of the neighborhood is given by $x_2 = 1 - x_1$. This is a straight line which intercepts the x_2 axis at 1, and has slope -1. By symmetry, it follows that the neighborhood is diamond, with corners at $(0,1)$, $(0,1)$, $(0,-1)$, and $(-1,0)$. It is plotted in Fig. 1.

10. Consider a function f defined on $\mathbb{R}$ which satisfies

$$|f(x) - f(y)| \leq (x-y)^2$$

for all $x, y \in \mathbb{R}$. Prove that f is a constant function. Suppose that f is not constant. Then there exists $x < y$ such that $f(x) \neq f(y)$. For an arbitrary $n \in \mathbb{N}$ consider the points

$$t_k = x - \frac{(y-x)k}{n}$$

for $k = 0, \ldots, n$. Then by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \\ &\leq \sum_{k=1}^{n} (t_k - t_{k-1})^2 \\ &= \sum_{k=1}^{n} \left(\frac{y-x}{n}\right)^2 \\ &= \frac{(y-x)^2}{n}. \end{aligned}$$

Since n is arbitrary, there exists an n such that

$$n > \frac{(y-x)^2}{|f(x) - f(y)|}.$$

which leads to a contradiction. An alternative approach is to first note that for all $x, y \in \mathbb{R}$

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq \left|\frac{(y-x)^2}{y-x}\right| = |y - x|.$$

Since $|y-x| \to 0$ as $y \to x$, it follows that

$$\left| \frac{f(y)-f(x)}{y-x} \right|$$

as $y \to x$, and hence

$$\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = 0$$

for all $x \in \mathbb{R}$. Thus f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$. For any bounded interval $I \subseteq \mathbb{R}$, f must be constant. Hence f is constant on $\mathbb{R}$.

11. Suppose that f is differentiable on $\mathbb{R}$, and that $2 \leq f'(x) \leq 3$ for $x \in \mathbb{R}$. If $f(0) = 0$, prove that $2x \leq f(x) \leq 3x$ for all $x \geq 0$. Suppose $x > 0$. By the mean value theorem, there exists a $y \in (0, x)$ such that

$$\frac{f(x)-f(0)}{x-0} = f'(y)$$

and hence

$$\frac{f(x)}{x} = f'(y).$$

The constraint on the derivative shows that

$$2 \leq \frac{f(x)}{x} \leq 3$$

and hence

$$2x \leq f(x) \leq 3x. \tag{1}$$

Since $f(0) = 0$, it follows that Eq. 1 holds for all $x \geq 0$.

12. Suppose that f is integrable on $[a, b]$. Then for all $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Consider the partition Q composed of the values a, c, d, and b; in general this will have four points, but if $c = a$ or $d = b$ it may have fewer. Define a new partition $T = P \cup Q$. Then since T is a refinement,

$$U(f, T) - L(f, T) < \epsilon.$$

Write $T = \{a = t_0 < t_1 < \ldots < t_n = b\}$. There exists i and j such that $t_i = c$ and $t_j = d$. Consider the partition $S = \{t_i < \ldots < t_j\}$ of $[c, d]$. Then

$$\begin{aligned}
U(f,S) - L(f,S) &= \sum_{k=i+1}^{j} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])) \\
&\leq \sum_{k=1}^{n} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])) \\
&= U(f,T) - L(f,T) \\
&< \epsilon.
\end{aligned}$$

Since this is true for an arbitrary $\epsilon > 0$, it follows that f is integrable on $[c, d]$.

13. (a) Suppose that $r^{1/3}$ is rational. Then

$$r^{1/3} = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$. Hence

$$r = \frac{p^3}{q^3}$$

and thus r is rational. Thus is r is irrational, then $r^{1/3}$ is irrational also. Similarly, if $r + 1$ is rational, then

$$r + 1 = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$, and

$$r = \frac{p}{q} - 1 = \frac{p - q}{q}$$

so r is rational. Thus if r is irrational, then $r + 1$ is irrational also.

(b) First, consider the number $x = 5 + \sqrt{2}$. Then

$$\begin{aligned} x - 5 &= \sqrt{2} \\ (x-5)^2 &= 2 \\ x^2 - 10x + 25 &= 2 \\ x^2 - 10x + 23 &= 0. \end{aligned}$$

By the rational zeroes theorem, if x is rational, then $x = \pm 1$ or $x = \pm 23$. But $x > 5$ and $x < 5 + 2 = 7$, so none of these possibilities are valid, and thus x is irrational. By the results of part (a), $(5 + \sqrt{2})^{1/3}$ is irrational, and $(5 + \sqrt{2})^{1/3} + 1$ is irrational also.

14. For the first limit, L'Hôpital's rule can be applied once to show that

$$\lim_{x \to 0} \frac{x}{1 - e^{-x^2 - 3x}} = \lim_{x \to 0} \frac{1}{(2x+3)e^{-x^2-3x}} = \frac{1}{3}.$$

The second limit can first be rewritten as

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}$$

after which L'Hôpital's rule can be applied twice to show that

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\sin x}{2 \cos x + x \sin x} = \frac{0}{2} = 0.$$

For the third limit, L'Hôpital's rule can be applied three times to show that

$$\begin{aligned}
\lim_{x\to 0} \frac{x^3}{\sin x - x} &= \lim_{x\to 0} \frac{3x^2}{\cos x - 1} \\
&= \lim_{x\to 0} \frac{6x}{-\sin x} \\
&= \lim_{x\to 0} \frac{6}{-\cos x} \\
&= \frac{6}{-1} = -6.
\end{aligned}$$

15. Suppose $a > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-2N} < a$. Hence for $M > 2N$,

$$\sup\{s_n : n > M\} = a$$

and hence $\limsup s_n = a$. Now consider any $l > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-n} < l$ for $n > 2N$. Hence l is not a lower bound for the set $\{s_n : n > M\}$ for any M. However, since 0 is lower bound, it follows that it must be the greatest lower bound, and hence $\limsup s_n = 0$.

Now suppose $a \le 0$. Then $\inf\{s_n : n > M\} = a$ for all $M \in \mathbb{N}$, so $\liminf s_n = a$. Also,

$$\sup\{s_n : n > M\} = \begin{cases} 2^{-M-1} & \text{if } M \text{ is odd,} \\ 2^{-M-2} & \text{if } M \text{ is even.} \end{cases}$$

so $\limsup s_n = 0$.

If $a = 0$ then $\limsup s_n = \liminf s_n$ and hence the series converges with limit zero. Otherwise, $\limsup s_n \ne \liminf s_n$ and the series does not converge.

16. Consider any $\epsilon > 0$. Then

$$\begin{aligned}
|f(x_1, x_2) - f(0,0)| &= \left| \frac{1}{x_1^2 + x_2^2 + 1} - \frac{1}{1} \right| \\
&= \left| \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + 1} \right| \\
&\le |x_1^2 + x_2^2|.
\end{aligned}$$

Suppose $d((x_1, x_2), (0,0)) < \delta$ where $\delta = \sqrt{\epsilon}$. Then

$$|f(x_1, x_2) - f(0,0)| \le (d((x_1, x_2), (0,0)))^2 < \delta^2 = \epsilon.$$

Now consider the point $(0,1)$:

$$\begin{aligned}
|f(x_1,x_2)-f(0,1)| &= \left|\frac{1}{x_1^2+x_2^2+1}-\frac{1}{2}\right| \\
&= \left|\frac{x_1^2+x_2^2-1}{2(x_1^2+x_2^2+1)}\right| \\
&\leq \frac{|x_1^2+x_2^2-1|}{2} \\
&= \frac{|x_1^2+(x_2-1)(x_2+1)|}{2} \\
&\leq \frac{|x_1|^2+|x_2-1|\cdot|x_2+1|}{2}.
\end{aligned}$$

Suppose $d((x_1,x_2),(0,1))<1$, so that

$$x_1^2+(x_2-1)^2<1.$$

from which it follows that $|x_1|<1$ and $|x_2-1|<1$. Hence $|x_2+1|\leq|x_2|+1\leq 3$. Define $\delta=\min\{1,\frac{\epsilon}{3}\}$. Then $\delta<\frac{\epsilon}{3}$ and $\delta<\sqrt{\epsilon}$. Hence $|x_1|<\sqrt{\epsilon}$ and $|x_2-1|<\frac{\epsilon}{3}$. Then

$$|f(x_1,x_2)-f(0,1)|\leq\frac{|x_1|^2+|x_2-1|\cdot|x_2+1|}{2}<\frac{(\sqrt{\epsilon})^2+3\frac{\epsilon}{3}}{2}=\epsilon.$$

Hence f is continuous at $(0,1)$.

17. (a) For $p\neq 1$, the improper integral can be written as

$$\begin{aligned}
\int_0^1 x^{-p}\,dx &= \lim_{c\to 0^+}\int_c^1 x^{-p}\,dx \\
&= \lim_{c\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_c^1 \\
&= \lim_{c\to 0^+}\frac{1-c^{1-p}}{1-p}.
\end{aligned}$$

Then if $0<p<1$, the exponent is positive, so

$$\int_0^1 x^{-p}\,dx=\frac{1}{1-p}$$

whereas if $p>1$, the exponent is negative, so

$$\int_0^1 x^{-p}\,dx=\infty.$$

(b) The improper integral can be written as

$$\int_0^1 x^{-p}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-p}\,dx + \lim_{d\to\infty}\int_1^d x^{-p}\,dx.$$

Since both terms are non-negative, it follows that if one term is ∞, then the integral must ∞ also. By the result above, if $p > 1$, then the first term is ∞. Now suppose $p = 1$. Then the second term is

$$\begin{aligned}\lim_{d\to\infty}\int_1^d x^{-1}\,dx &= \lim_{d\to\infty}[\log x]_1^d \\ &= \lim_{d\to\infty}\log d \\ &= \infty.\end{aligned}$$

If $0 < p < 1$, then second term is

$$\begin{aligned}\lim_{d\to\infty}\int_1^d x^{-p}\,dx &= \lim_{d\to\infty}\left[\frac{x^{-p+1}}{1-p}\right]_1^d \\ &= \lim_{d\to\infty}\frac{d^{-p+1}-1}{1-p} \\ &= \infty.\end{aligned}$$

Hence in all cases, $\int_0^1 x^{-p}dx = \infty$.

18. Since f is integrable on $[a,b]$, it is bounded, so there exists a $B > 0$ such that $f(x) < B$ for all $x \in [a,b]$. Assume that if f is integrable on $[a,b]$ then it is integrable on any interval $[c,d] \subseteq [a,b]$; for full details see Problem 12.

To show the above limit, consider any $\epsilon > 0$, and examine $c \in (b-\delta, b)$ where $\delta = \epsilon/B$. Then

$$\begin{aligned}\left|\int_a^c f - \int_a^b f\right| &= \left|\int_c^b f\right| \\ &\le \int_c^b |f| \\ &\le (b-c)B \\ &< \delta B = \epsilon.\end{aligned}$$

Hence

$$\lim_{d\to b^-}\int_a^d f(x)\,dx = \int_a^b f(x)\,dx.$$

19. If $\lambda = 0$, then $s_n = 0$ for all n, so s_n is a constant convergent sequence. Similarly if $\lambda = 1$, then $s_n = 1$ for all n, so s_n is also constant and convergent. In general,

$$s_n = \lambda^{2^{n-1}}.$$

If $0 < \lambda < 1$, then since $0 < \lambda^{2^{n-1}} \leq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \to 0$ as $n \to \infty$ it follows that $s_n \to 0$. If $\lambda > 1$, then since $\lambda^{2^{n-1}} \geq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \to \infty$ as $n \to \infty$, it follows that $s_n \to \infty$.

Finally, suppose $\lambda < 0$. Then for $n \geq 2$, the resulting sequence (s_n) will be the same as the case for $-\lambda$, and will therefore have the same convergence properties. Hence (s_n) converges if and only if $|\lambda| \leq 1$.

20. The minimum element of $[0, \sqrt{2}]$ is 0, and since this is also in A it follows that $\min A = 0$. The maximum element of $[0, \sqrt{2}]$ is $\sqrt{2}$, but this is not in A. Since there are rational numbers arbitrarily close to $\sqrt{2}$, it follows that A does not have a maximum. The infimum is just $\inf A = \min A = 0$. $\sqrt{2}$ is an upper bound for A. For any $\epsilon > 0$, there exist elements in A which are greater than $\sqrt{2} - \epsilon$, and thus $\sqrt{2}$ is the least upper bound. Hence $\sup A = \sqrt{2}$.

For B, note that

$$x^2 + x - 1 = (x + 1/2)^2 - 5/4.$$

Since the first term can take any positive value, it follows that $B = [-5/4, \infty)$. Hence $\min B = \inf B = -5/4$, the maximum does not exist, and $\sup B = \infty$.

By completing the square, above equation can be written as

$$x^2 + x - 1 = \left(x + \frac{1 - \sqrt{5}}{2} \right) \left(x + \frac{1 + \sqrt{5}}{2} \right).$$

The quadratic will be strictly negative when one of these two factors is strictly negative and the other is strictly positive. Hence $C = (-1/2 - \sqrt{5}/2, -1/2 + \sqrt{5}/2)$, so the minimum and maximum do not exist, $\inf C = -1/2 - \sqrt{5}/2$ and $\sup C = -1/2 + \sqrt{5}/2$.

21. (a) Suppose $0 \leq x < 1$. Then

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (x - x^n) = x$$

since if $|x| < 1$, then $x^n \to 0$ as $n \to \infty$. For $x = 1$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (1 - 1^n) = 0.$$

Hence f_n converges pointwise to a limit f on $[0, 1]$ given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Graphs of several of the f_n and the limit f are shown in Fig. 2.

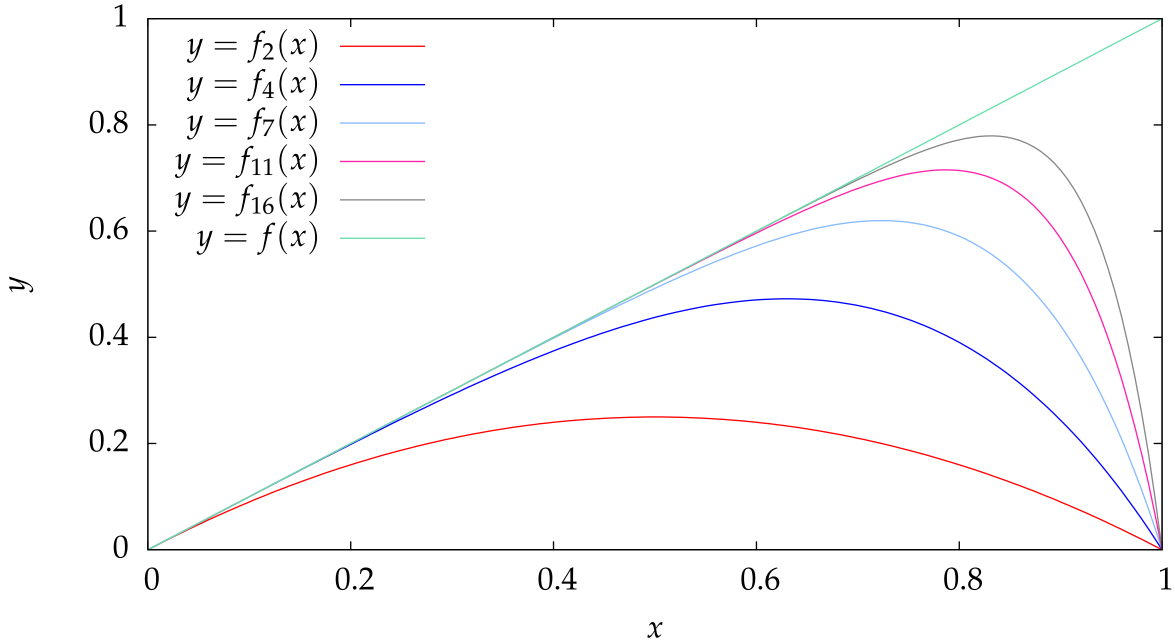

Figure 2: Graph for Problem 21, on pointwise and uniform convergence.

(b) As can be seen from Fig. 2, the convergence does not appear to be uniform on $[0, 1]$, since it does not appear that the f_n will ever lie within a strip of a fixed width ϵ around f. To see this mathematically, for a given n, consider the point $x = (1/2)^{1/n}$. Then

$$\begin{aligned} |f_n(x) - f(x)| &= |x - x^n - x| \\ &= |x^n| \\ &= 1/2 \end{aligned}$$

Hence if $\epsilon = 1/2$, there does not exist an N such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$.

(c) Consider the interval $[0, 1/2]$. Then for any x in this interval

$$|f_n(x) - f(x)| = |x^n| \leq 2^{-n}$$

Consider any $\epsilon > 0$. Then there exists an N such that $n > N$ implies $2^{-n} < \epsilon$, and thus $|f_n(x) - f(x)| < \epsilon$. Hence $f_n \to f$ uniformly on $[0, 1/2]$.

(d) Since continuous functions are integrable, it follows immediately that f_n is integrable for all $n \in \mathbb{N}$. For a specific n,

$$\begin{aligned}\int_0^1 f_n &= \int_0^1 (x - x^n)dx \\ &= \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{n+1} \\ &= \frac{n-1}{2(n+1)}.\end{aligned}$$

To show that f is integrable, consider the function

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Choose any $\epsilon > 0$, and examine the partition $P = \{0 = t_0 < t_1 < t_2 = 1\}$ where $t_1 = 1 - \epsilon/2$. Then

$$L(f,P) = \sum_{k=1}^{2} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = \sum_{k=1}^{2} 0 = 0$$

and

$$U(f,P) = \sum_{k=1}^{2} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = (t_1 - t_0) \cdot 0 + (t_2 - t_1)1 = \frac{\epsilon}{2}.$$

Thus $U(f,P) - L(f,P) < \epsilon$. Since a partition such as this can be constructed for an arbitrary $\epsilon > 0$, it follows that g is integrable and $\int_0^1 g = 0$. Since $f(x) = x - g(x)$ and both x and g are integrable, it follows that f is integrable and $\int_0^1 f = \int_0^1 x - \int_0^1 g = 1/2$. Note that

$$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \frac{n-1}{2(n+1)} = \frac{1}{2} = \int_0^1 f.$$

22. (a) If $0 \leq x \leq 1$, then

$$F(x) = \int_0^x f = \int_0^x 1\, dt = x.$$

If $1 < x \leq 2$, then

$$F(x) = \int_0^1 f + \int_1^x = 1 - 2(x-1) = 3 - 2x.$$

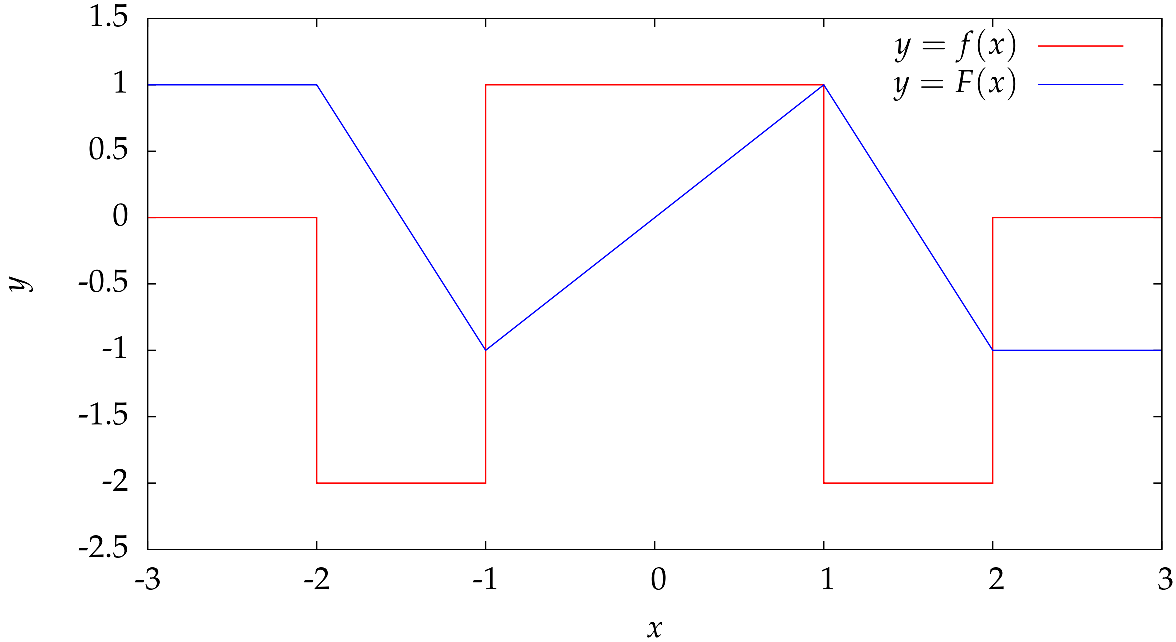

Figure 3: Graph for Problem 22, on the second Fundamental Theorem of Calculus.

If $x > 2$, then

$$F(x) = F(2) + \int_2^x 0 = F(2) = -1.$$

For negative values of x, note that f is an even function, and thus

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^x f(-s)(-ds) = -\int_0^x f(s)ds = -F(x)$$

so F is odd. Hence

$$F(x) = \begin{cases} 1 & \text{if } x < -2, \\ -3 - 2x & \text{if } -2 \le x < 1, \\ x & \text{if } -1 \le x \le 1, \\ 3 - 2x & \text{if } 1 < x \le 2, \\ -1 & \text{if } x > 2. \end{cases}$$

(b) The functions f and F are plotted in Fig. 3.

(c) By the second Fundamental Theorem of Calculus, if f is continuous at x, and then F is differentiable at x and $F'(x) = f(x)$. Thus the only points where F may not be defined are $x = \pm 1, \pm 2$. Since

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = 1$$

but

$$\lim_{x\to 1^+} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^-} \frac{3-2x-1}{x-1} = -2$$

so F is not differentiable at 1. Similarly

$$\lim_{x\to 2^-} \frac{F(x)-F(2)}{x-2} = \lim_{x\to 2^-} \frac{3-2x-(-1)}{x-2} = 1$$

but

$$\lim_{x\to 2^+} \frac{F(x)-F(2)}{x-2} = \lim_{x\to 2^+} \frac{0}{x-2} = 0$$

so F is not differentiable at 2. Since F is odd, it follows that F is not differentiable at -1 and -2 also. Hence F' is defined on $\mathbb{R}/\{-2,-1,1,2\}$ and

$$F(x) = \begin{cases} 0 & \text{if } x < -2, \\ -2 & \text{if } -2 < x < 1, \\ 1 & \text{if } -1 < x < 1, \\ -2 & \text{if } 1 < x < 2, \\ 0 & \text{if } x > 2. \end{cases}$$

23. (a) Let f and g be continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists an $x \in [a,b]$ such that $f(x) = g(x)$. If $\int_a^b f = \int_a^b g$, then if $h(x) = f(x) - g(x)$, then $\int_a^b h = 0$. Consider the partition $P = \{a = t_0 < t_1 = b\}$. Then

$$0 = \int_a^b h \le U(h,P) = (b-a)M(h,[a,b])$$

and

$$0 = \int_a^b h \ge L(h,P) = (b-a)m(h,[a,b]).$$

Since a continuous function on a closed interval achieves its bounds, there exist x_1 and x_2 such that $h(x_1) = M(h,[a,b])$ and $h(x_2) = m(h,[a,b])$. Either $h(x_1) = 0$ or $h(x_2) = 0$, or otherwise $h(x_1) > 0$ and $h(x_2) < 0$. In the latter case, the intermediate value theorem can be applied to show that there exists an x_3 between x_1 and x_2 such that $h(x_3) = 0$. In all cases there exists an x such that $h(x) = 0$ and hence $f(x) = g(x)$.

(b) On the interval $[-1,1]$, define

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \ge 0, \end{cases}$$

and let $g(x) = -f(x)$. By construction $f(x) \ne g(x)$ for all $x \in [-1,1]$. To find the integral of f, choose $\epsilon > 0$ and consider the partition $P = \{a = -1 < t_1 <$

$t_2 < t_3 = 1\}$ where $t_1 = -\epsilon/5$ and $t_2 = \epsilon/5$. Then

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{3} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(-1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= -\frac{2\epsilon}{5}. \end{aligned}$$

Similarly

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{3} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= \frac{2\epsilon}{5}. \end{aligned}$$

Then $U(f, P) - L(f, P) = 4\epsilon/5 < \epsilon$, and since ϵ is arbitrary it follows that f is integrable, and that $\int_{-1}^{1} f = 0$. In addition, so $\int_{-1}^{1} g = \int_{-1}^{1} (-f) = -\int_{-1}^{1} f = 0$. Thus $\int_{-1}^{1} f = \int_{-1}^{1} g$ but $f(x) \neq g(x)$ for all $x \in [-1, 1]$.

24. (a) The functions h_1, h_2, and h_3 are plotted in Fig. 4

(b) Consider any $x \neq 0$, and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $1/N < 2|x|$. Then if $n > N$, $h_n(x) = 0$. Hence $\lim_{n\to\infty} h_n(x) = 0$. At $x = 0$,

$$h_n(x) = n$$

which tends to ∞ as $n \to \infty$.

(c) Consider any $\epsilon > 0$. Then since f is continuous, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon$. Then there exists an N such that $1/2N < \delta$. For $n > N$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n f.$$

(d) Consider

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then for any $n \in \mathbb{N}$,

$$\int_{\infty}^{\infty} h_n g = \int_{-1/2n}^{1/2n} ng = \int_{-1/2n}^{0} 0 + \int_{0}^{1/2n} n = \frac{1}{2}$$

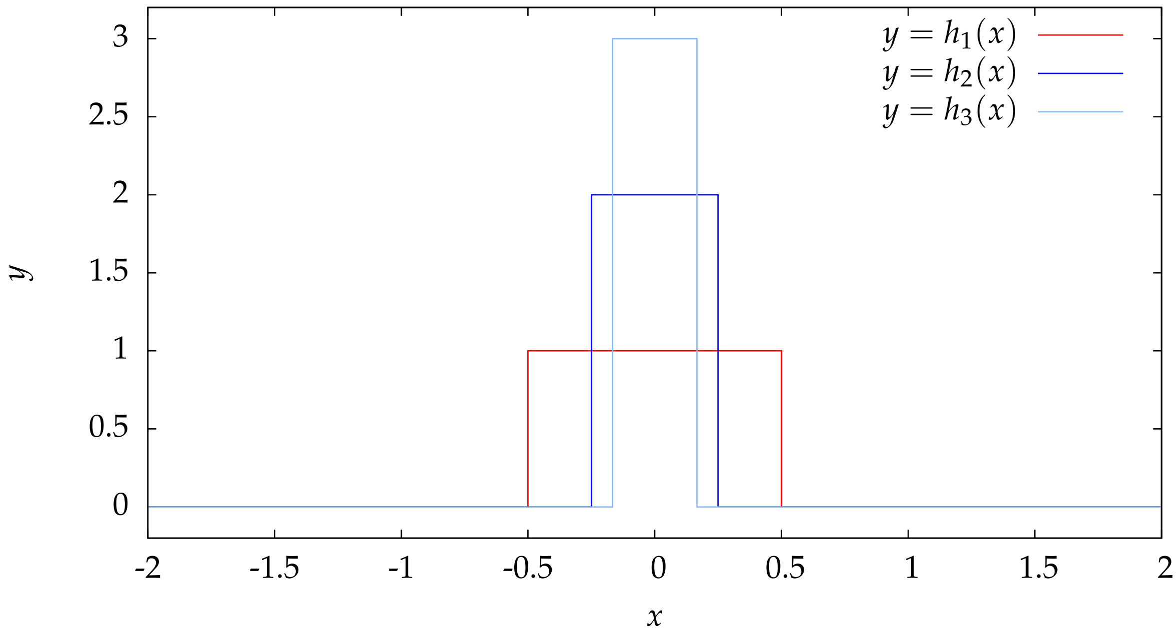

Figure 4: Graphs of several functions $h_n(x)$ used in Problem 24 on integration limits.

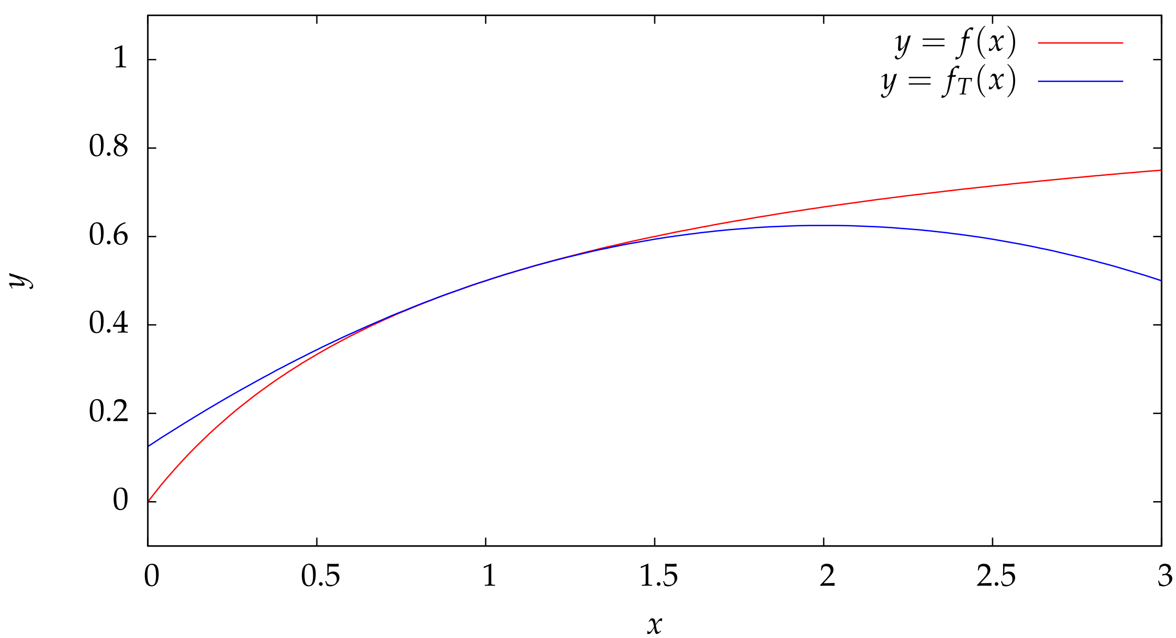

Figure 5: A graph of the function f and a Taylor series approximation f_T at $x = 1$, discussed in Problem 25.

and hence

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h_n g = \frac{1}{2}$$

which does not equal $g(0) = 1$.

25. (a) For $x > 0$,

$$f(x) = \frac{1}{1+\frac{1}{x}}$$

and since $1/x \to 0$ as $x \to \infty$, it follows that $f(x) \to 0$ as $x \to \infty$. On the interval under consideration $0 \le x < 1 + x$, so $0 \le f(x) < 1$.

(b) The function f is shown in Fig. 5.

(c) Since

$$f(x) = 1 - \frac{1}{x+1},$$

it follows that the derivatives are

$$f'(x) = \frac{1}{(x+1)^2}$$

and

$$f''(x) = -\frac{2}{(x+1)^3}.$$

Hence $f(1) = 1/2$, $f'(1) = 1/4$, $f''(1) = -1/4$ and thus

$$\begin{aligned} f_T(x) &= \sum_{n=0}^{2} \frac{(x-1)^n f^{(n)}(1)}{n!} \\ &= f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2} \\ &= \frac{1}{2} + \frac{x-1}{4} - \frac{(x-1)^2}{8}. \end{aligned}$$

(d) f_T can rewritten as

$$\begin{aligned} f_T(x) &= \frac{1}{2} + \frac{x}{4} - \frac{1}{4} - \frac{x^2}{8} + \frac{x}{4} - \frac{1}{8}. \\ &= \frac{1}{8} + \frac{x}{2} - \frac{x^2}{8} \end{aligned}$$

which is a quadratic.

(e) The function f_T is shown in Fig. 5. By construction, the curves intersect at $x = 1$, and have the same slope and curvature there.