Solutions to sample midterm questions

1. Let \( x = 1 + \sqrt{1 + \sqrt{2}} \). Then

\[
\begin{align*}
    x - 1 &= \sqrt{1 + \sqrt{2}} \\
    (x - 1)^2 &= 1 + \sqrt{2} \\
    x^2 - 2x + 1 &= 1 + \sqrt{2} \\
    x^2 - 2x &= \sqrt{2} \\
    x^4 - 4x^3 + 4x^2 &= 2 \\
    x^4 - 4x^3 + 4x^2 - 2 &= 0
\end{align*}
\]

Hence if \( x = p/q \), then \( p \) divides 2 and \( q \) divides 1. The only possibilities are \( \pm 1 \), and \( \pm 2 \). But \( \sqrt{1 + \sqrt{2}} > 1 \), and thus \( x > 2 \). Thus \( x \) must be irrational.

2. Define \( a_n = 8^n/(n!)^2 \). Then

\[
\frac{a_{n+1}}{a_n} = \frac{8^{n+1}}{(n+1)^2} \cdot \frac{(n!)^2}{8^n} = \frac{8}{(n+1)^2} \to 0
\]

and therefore by the ratio test, \( \sum 8^n/(n!)^2 \) converges.

Now consider \( \sum (-1)^n b_n \) where \( b_n = 1/\sqrt{n^2 + n} \). Since \( n \) and \( n^2 \) are both increasing functions, \( n^2 + n \) is an increasing function also, and hence \( 1/\sqrt{n^2 + n} \) is a decreasing function. In addition, \( b_n < 1/n \) for all \( n \), so \( b_n \to 0 \) as \( n \to \infty \). Therefore, the series \( \sum (-1)^n b_n \) satisfies the conditions for the alternating series theorem, and hence it converges.

3. (a) Let \( x \in S \cup T \). Then either \( x \in S \) so \( x \leq \sup S \), or \( x \in T \) so \( x \leq \sup T \). Hence, \( x \leq \max \{ \sup S, \sup T \} \). Thus \( \max \{ \sup S, \sup T \} \) is an upper bound for \( \sup S \cup T \).

Now suppose that \( m \) is an upper bound for \( S \cup T \). Hence \( m \geq x \) for all \( x \in S \cup T \). Thus \( m \geq s \) for all \( s \in S \), so \( m \geq \sup S \) as \( \sup S \) is the least upper bound for \( S \). Similarly \( m \geq t \) for all \( t \in T \). Hence \( m \geq \sup T \) as \( \sup T \) is the least upper bound of \( T \). Therefore \( m \geq \max \{ \sup S, \sup T \} \). Hence \( \max \{ \sup S, \sup T \} \) is an upper bound, and it is the least upper bound, so it must equal \( \sup S \cup T \).

Now consider \( x \in S \cap T \). Hence \( x \in S \) and \( x \in T \). Then \( x \leq \sup S \) and \( x \leq \sup T \), so \( x \leq \min \{ \sup S, \sup T \} \), and therefore \( \sup S \cap T \leq \min \{ \sup S, \sup T \} \).

(b) For a non-empty set \( A \), \( \sup A \neq -\infty \), so it suffices to consider when the suprema become positive infinity. Suppose \( \sup S = \infty \). Then \( S \) is not bounded
above. Hence \( S \cup T \) is not bounded above. Therefore \( \sup S \cup T = \infty \) and the identity still holds.

For the second identity, if \( \sup S = \infty \), then \( \min\{\sup S, \sup T\} = \sup T \). Since \( \sup T \) is an upper bound for \( T \), it is also an upper bound for \( S \cap T \), and hence the identity still holds.

The same arguments can be applied if \( \sup T = \infty \).

(c) Consider \( S = \{1, 3\} \) and \( T = \{1, 2\} \). Then \( \sup S = 3 \) and \( \sup T = 2 \), so \( \min\{\sup S, \sup T\} = 2 \). However, \( S \cap T = \{1\} \) and so \( \sup S \cap T = 1 < 2 \).

4. Let \( \lim s_n = s \). Since \( s_n \) converges, there exists an \( N_1 \) such that \( n > N_1 \) implies that \( |s_n - s| < 1 \). Hence \(-1 < s_n - s \) and \( s_n > s - 1 \).

Now pick \( M > 0 \). Since \( t_n \) diverges, there exists an \( N_2 \) such that

\[
t_n > 1 - s + M
\]

for all \( n > N_2 \). Hence for \( n > \max\{N_1, N_2\} \),

\[
s_n + t_n > (s - 1) + 1 - s + M = M
\]

and thus \( s_n + t_n \) diverges to infinity.

5. (a) Define \( a_N = \sup\{s_n : n > N\} \) and \( b_N = \sup\{t_n : n > N\} \). Now, for \( n > N \),

\[
s_n + t_n \leq a_N + b_N
\]

since \( a_N \) and \( b_N \) are upper bounds for \( s_n \) and \( t_n \). If \( c_N = \sup\{s_n + t_n : n > N\} \), then

\[
c_N \leq a_N + b_N.
\]

The sequences \( (a_N) \), \( (b_N) \), and \( (c_N) \) are non-increasing. Suppose that \( \lim c_N > \lim a_N + \lim b_N \). Then \( \lim c_N = \lim a_N + \lim b_N + \epsilon \) for some \( \epsilon > 0 \), so there exist \( K_1 \) and \( K_2 \) such that if \( k > K_1 \)

\[
ak < \lim a_N + \frac{\epsilon}{3}
\]

and if \( k > K_2 \) then

\[
bk < \lim b_N + \frac{\epsilon}{3}.
\]

Now for all \( k > \max\{K_1, K_2\} \),

\[
c_k \leq a_k + b_k
\]

\[
< \left( \lim a_N + \frac{\epsilon}{3} \right) + \left( \lim b_N + \frac{\epsilon}{3} \right)
\]

\[
< (\lim a_N + \lim b_N + \epsilon) - \frac{\epsilon}{3} = \lim c_N - \frac{\epsilon}{3}.
\]

But then \( |c_k - \lim c_N| > \frac{\epsilon}{3} \) for all \( k > \max\{K_1, K_2\} \), so \( c_k \) does not converge to \( \lim c_N \) which is a contradiction. Hence \( \lim c_N \leq \lim a_N + \lim b_N \), and hence \( \lim \sup s_n + t_n \leq \lim \sup s_n + \lim \sup t_n \).
(b) Suppose
\[ s_n = \begin{cases} 
1 & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases} \]
and that
\[ t_n = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd}.
\end{cases} \]

Then \( \sup\{s_n : n > N\} \) and \( \sup\{t_n : n > N\} = 1 \) for all \( N \in \mathbb{N} \), and hence
\[
(\lim \sup s_n) \cdot (\lim \sup t_n) = 1 \cdot 1 = 1.
\]

However, \( s_n t_n = 0 \) for all \( n \), and thus \( \lim \sup(s_n t_n) = 0 \neq 1 \).

6. Suppose that \( a > b \). Then define \( \epsilon = a - b > 0 \). Then there exists an \( N_1 \) such that \( n > N_1 \) implies that \( |a_n - a| < \epsilon/2 \). Similarly there exists an \( N_2 \) such that \( n > N_2 \) implies that \( |b_n - b| < \epsilon/2 \). Now consider any \( k \) such that \( k > \max\{N_1, N_2\} \). Then \( |a_k - a| < \epsilon/2 \), and hence \( -\epsilon/2 < a_k - a \), so
\[
a_k > a - \frac{\epsilon}{2} = a - \frac{a - b}{2} = \frac{a + b}{2}.
\]

In addition, \( |b_k - b| < \epsilon/2 \), so \( b_k - b < \epsilon/2 \), and hence
\[
b_k < b + \frac{\epsilon}{2} = b + \frac{a - b}{2} = \frac{a + b}{2}.
\]

Combining these two inequalities shows that \( a_k > b_k \), which is a contradiction. Thus \( a \leq b \).

7. Since the lower limit of \( A \) is an open interval, it does not have a minimum, however \( \inf A = 0 \). Since \( A \) is not bounded above, it does not have a maximum. \( \sup A = \infty \) for sets not bounded above.

Since \( B \) has no smallest element, the minimum does not exist. However, since the fractions become arbitrarily close to 0, \( \inf B = 0 \). The maximum is given by \( \max B = 1/2 \), attained for the case when \( n = 1 \), and hence \( \sup B = \max B = 1/2 \).

8. For the first sequence, make use of the root test where \( a_n = 6^n/n^n \). Then
\[
(a_n)^{1/n} = \frac{6}{n}
\]
which converges to zero as \( n \to \infty \). Hence \( \sum 6^n/n^n \) converges. For the second sequence, since \( n + 1/2 \leq 2n \) for all \( n \in \mathbb{N} \), then
\[
\frac{1}{n + 1/2} \geq \frac{1}{2n}
\]
for all \( n \in \mathbb{N} \). Since \( \sum \frac{1}{n} \) diverges, so does \( \sum \frac{1}{2n} \), and hence by the comparison test, \( \sum 1/(n + 1/2) \) does also.
9. Choose an element \( t \in T \). Then either
   
   - \( t \in S \). Hence \( t \leq \sup S \).
   
   - There exists \( s \in S \) such that \( s = -t \). Hence \( s \geq \inf S \), and therefore \( t \leq -\inf S \).

Thus either \( t \leq \sup S \) or \( t \leq -\inf S \) so \( t \leq \max\{\sup S, -\inf S\} \). Hence it is an upper bound.

Now suppose that \( l \) is an upper bound for \( T \). Then \( l \geq t \) for all elements \( t \in T \). Hence \( l \geq |s| \) for all elements \( s \in S \), and thus

\[
-1 \leq s \leq 1
\]

for all elements in \( s \), from which the following two deductions can be made:

   - Since \( s \leq l \) for all \( s \), then \( l \geq \sup S \) since \( \sup S \) is the least upper bound for \( S \).
   
   - Since \( -1 \leq s \) for all \( s \), then \( -1 \leq \inf S \) since \( \inf S \) is the greatest lower bound for \( S \). Hence \( l \geq -\inf S \).

These two results show that \( l \geq \max\{\sup S, -\inf S\} \). Hence \( \max\{\sup S, -\inf S\} \) is an upper bound for \( T \) and it is the least upper bound, so it must be \( \sup T \).

10. Define \( v_N = \sup\{s_n : n > N\} \). There are two cases:

   - \( \limsup t_n = -q \) for some \( q > 0 \). Then there exists a \( K_1 \) such that \( |v_N - (-q)| < q/2 \) for all \( N > K_1 \). Hence \( v_{K_1+1} < (-q) + (q/2) = -q/2 \), and thus \( t_n < -q/2 \) for all \( n > K_1 + 1 \). For this case, define \( \lambda = -q/2 \).

   - \( \limsup t_n = -\infty \). Then there exists a \( K_1 \) such that \( v_N < -1 \) for all \( N > K_1 \). Hence \( v_{K_1+1} < -1 \), and thus \( t_n < -1 \) for all \( n > K_1 + 1 \). For this case, define \( \lambda = -1 \).

Now consider the sequence \( s_nt_n \). Pick \( M < 0 \). Then since \( \lim s_n = \infty \), there exists a \( K_2 \) such that \( n > K_2 \) implies that \( s_n > M/\lambda \).

Now suppose \( n > \max\{K_1 + 1, K_2\} \). Then \( s_n > M/\lambda \) and \( t_n < \lambda \), so \( s_nt_n < M \). This is true for any \( M < 0 \), so \( \lim s_nt_n = -\infty \).