The wonder of \( \pi \)

Amherst Town Library
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The circle

Area = \( \pi r^2 \)

Circumference = \( 2\pi r \)
The wonder of pi

- The calculation of pi is perhaps the only mathematical problem that has been of continuous interest from antiquity to present day.

- Many famous mathematicians throughout history have considered it, and as such it provides a window into the development of mathematical thought itself.
Assumed knowledge

- Many symbols and thought processes that we take for granted are the product of millennia of development.
- Early explorers of pi had no such tools available.
An early measurement

- Purchased by Henry Rhind in 1858, in Luxor, Egypt
- Scribed in 1650 BCE, and copied from an earlier work from ~2000 BCE
- One of the oldest mathematical texts in existence
- Consists of fifty worked problems, the last of which gives a value for $\pi$
Problem 24
(An example of Egyptian mathematical logic)

A heap and its 1/7 part become 19. What is the heap?

Then 1 heap is 7.
And 1/7 of the heap is 1.
Making a total of 8.

But this is not the right answer, and therefore we must rescale 7 by the proportion of 19/8 to give

\[ 7 \times \frac{19}{8} = 16 \frac{5}{8} \]

(Guess an answer and see if it works.)
(Re-scale the answer to obtain the correct solution.)
Problem 24

A heap and its 1/7 part become 19. What is the heap?

• Modern approach using high-school algebra: let $x$ be the size of the heap. Then

\[ x + \frac{x}{7} = 19 \]

• Rearranging gives

\[ \frac{8x}{7} = 19 \quad \Rightarrow \quad x = \frac{19 \times 7}{8} = 16\frac{5}{8} \]
Problem 50

- A circular field with diameter 9 units has the same area as a square with side eight units.
- Compare areas

\[ \pi \left( \frac{9}{2} \right)^2 = 8^2 \]

- Gives a value for \( \pi \)

\[ \pi = 4 \left( \frac{8}{9} \right)^2 = 3.16049 \ldots \]
Octagon method?

- Egyptians made use of square nets
- Cover circle in a 3 by 3 grid
- Area of the octagon is 63, which gives

\[
\pi = 4 \left( \frac{63}{81} \right) \approx 4 \left( \frac{64}{81} \right) = 4 \left( \frac{8}{9} \right)^2
\]
Other values from antiquity

- Babylonians: \( \pi = 3 \ 1/8 \)
- From the bible:

  And he made the molten sea of ten cubits from brim to brim, round in compass, and the height thereof was five cubits; and a line of thirty cubits did compass it round about.

  – 1 Kings 7:23

- Implies \( \pi = 3 \), although much debate about exact meaning
Archimedes (278BCE–212BCE)

- Brilliant physicist, engineer, and mathematician
- First to realize that the same constant appears in $C = 2\pi r$ and $A = \pi r^2$
- Sandwiches circle between inscribed and superscribed polygons
Archimedes’ method

• Know that

\[
\text{Circumference of inscribed hexagon} < \pi < \text{Circumference of superscribed hexagon}
\]

• By repeatedly doubling the number of polygon sides from 6, 12, 24, 48, to 96, the calculation becomes more accurate

• Archimedes obtains

\[
\frac{10}{71} < \pi < \frac{1}{7}
\]
Developments in Asia
(during the first millennium)

• Religious persecution of science brings study of $\pi$ to a halt in Europe

• Most developments in Asia, including decimal notation

• Zu Chongzhi (429–500) uses a polygon with $3 \times 2^{14}$ sides; obtains $\pi = 3.1415926...$

• Also finds ratio $\pi \approx 355/113$

• Best result for a millennium!

Zu Chongzhi postage stamp
European middle ages (1000–1500)

- Figures of 3 1/8, 22/7, and (16/9)^2 still in use
- Decimal notation gradually introduced from the east
- Leonardo of Pisa (Fibonacci) uses Archimedes’ method, but more accurately; obtains

\[
\pi = \frac{864}{275} = 3.141818\ldots
\]
François Viète (1540–1603)

- Made important contributions to algebra, unifying geometrical ideas from Greeks, and numerical ideas from Arabs
- Using Archimedes’ approach, he finds the first infinite series for pi

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}} \cdots
\]

- While unwieldy, it can in principle be used to calculate pi to arbitrary precision
Viète series

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2}}
\]

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}}
\]

\[
\frac{2}{\pi} = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}}}
\]

\[
\pi = 2.828427\ldots
\]

\[
\pi = 3.061467\ldots
\]

\[
\pi = 3.121445\ldots
\]

By taking more terms, the infinite series converges to \(\pi\)
James Gregory (1638–1675)

- Scottish mathematician and astronomer
- Discovers arctangent formula, which forms the foundation of much better methods to calculate pi

(Wikipedia)
Circles and degrees

- The purple line is 30 degrees from the horizontal
- 360 degrees make a complete rotation
Circles and radians

- Mathematicians prefer to measure angles in radians (rad)
- $2\pi$ radians make a whole rotation
- Hence $180^\circ = \pi \text{ rad}$ and $30^\circ = \pi/6 \text{ rad}$
Why is this useful?

If we measure angles in radians, then the arc length of the circle covered by the angle is given by

\[ \frac{h\pi}{6} \]

This only works for radians, not degrees

\[ (\text{Circle radius}) \times (\text{Angle}) \]
A range of radians

We can draw out a whole range of angles. $\pi/4$ radians is $45^\circ$ and $\pi/2$ radians is $90^\circ$. 
The tangent of an angle is defined as

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Note that

$$\tan \frac{\pi}{4} = 1$$

since the Opposite and Adjacent are the same for a 45° triangle.
Arctangent

- The arctangent is the “inverse tangent”
- If $x = \tan y$, then the arctangent is written as $y = \tan^{-1} x$

- Since $\tan \frac{\pi}{4} = 1$ it follows that $\frac{\pi}{4} = \tan^{-1} 1$

- If we knew how to calculate an arctangent, this would let us determine $\pi$
Gregory's arctangent series

- Gregory shows that
  \[ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \]

- Put \( x = 1 \). Then since \( \frac{\pi}{4} = \tan^{-1} 1 \) we have
  \[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \]

- This is a much simpler formula but it converges very slowly. Calculating a thousand terms only gives three digits of pi
How to do much better

• However, Gregory’s formula is a lot better at computing arctangents of fractions. Putting $x = 1/2$ in his formula gives

$$\tan^{-1} \frac{1}{2} = \frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} + \ldots$$

• This converges much faster, but is that useful? What does $\tan^{-1} \frac{1}{2}$ have to do with pi?
How to do much better

- Could also calculate $\tan^{-1} \frac{1}{3}$ very efficiently using the same idea.
- Amazingly,
  \[
  \frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}
  \]
- Both terms here can be rapidly calculated, and it forms the basis of modern methods to calculate $\pi$. 
John Machin (1680–1752)

- English mathematician and astronomer who worked in London
- Derived a particularly efficient arctangent formula
  \[
  \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}
  \]
- Used it to calculate 100 digits of pi
William Jones (1675–1749)
(The first use of the symbol $\pi$ with its modern meaning)

Taking $a$ as an arc of 30°, and $t$ as a tangent in a figure given, he states (p. 243):

\[ 6a, \text{ or } 6 \times t - \frac{1}{3}t^2 + \frac{1}{5}t^5, \text{ &c.} = \frac{1}{2} \text{ Periphery (} \pi \text{)} \ldots \]

Let

\[ \alpha = 2\sqrt{3}, \beta = \frac{1}{3}\alpha, \gamma = \frac{1}{3}\beta, \delta = \frac{1}{3}\gamma, \text{ &c.} \]

Then

\[ \alpha - \frac{1}{3}\beta + \frac{1}{5}\gamma - \frac{1}{7}\delta + \frac{1}{9}\epsilon, \text{ &c.} = \frac{1}{2}\pi, \]

or

\[ \alpha - \frac{1}{3} \frac{3\alpha}{9} + \frac{1}{5} \frac{\alpha}{9} - \frac{1}{7} \frac{3\alpha}{9^2} + \frac{1}{9} \frac{\alpha}{9^2} - \frac{1}{11} \frac{3\alpha}{9^3} + \frac{1}{13} \frac{\alpha}{9^3}, \text{ &c.} \]

Thereof. the (Radius is to $\frac{1}{2}$ Periphery, or) Diameter is to the Periphery, as 1,000, &c to $3.141592653 \ . \ 5897932384 \ . \ 6264338327 \ . \ 9502884197 \ . \ 1693993751 \ . \ 0582097494 \ . \ 4592307816 \ . \ 4062862089 \ . \ 9862803482 \ . \ 5342117067$. 9+ True to above a 100 Places; as Computed by the accurate and Ready Pen of the Truly Ingenious Mr. John Machin.
There are various other ways of finding the Lengths, or Areas of particular Curve Lines, or Planes, which may very much facilitate the Practice; as for Instance, in the Circle, the Diameter is to Circumference as 1 to

\[
\frac{16}{3} - \frac{4}{239} - \frac{1}{3} \frac{16}{53} - \frac{4}{239^3} + \frac{1}{5} \frac{16}{5^6} - \frac{4}{239^6} - \ldots, \text{ &c.} = 3.14159, \text{ &c.} = \pi \ldots
\]

Whence in the Circle, any one of these three, \(\alpha, c, d\), being given, the other two are found, as, \(d = c \div \pi = \alpha \div \frac{1}{4\pi} \sqrt{\alpha}, c = d \times \pi = \alpha \times 4\pi \sqrt{\alpha}, \alpha = \frac{1}{4\pi} d^2 = c^2 \div 4\pi\).
Leonhard Euler (1707–1783)

He calculated just as men breathe, as eagles sustain themselves in the air

– François Arago

Now I will have less distraction

– Euler after losing his right eye

- Derived many infinite series such as

\[
\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots
\]

- Calculated 20 digits in an hour using

\[
\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}
\]
The arctangent digit hunters

1706: John Machin, 100 digits
1719: Thomas de Lagny, 112 digits
1739: Matsunaga Ryohitsu, 50 digits
1794: Georg von Vega, 140 digits
1844: Zacharias Dase, 200 digits
1847: Thomas Clausen, 248 digits
1853: William Rutherford, 440 digits
1876: William Shanks, 707 digits

Incorrect after 527 digits!
An ode to Shanks
(by Nicholas J. Rose)

Seven hundred seven
Shanks did state
Digits of pi he would calculate
And none can deny
It was a good try
But he erred at five twenty eight!
The nature of pi

- To understand numbers, mathematicians like to classify them into sets.
- Simplest starting point: the natural numbers: $1, 2, 3, 4, 5, 6, \ldots$
Further extensions

• Natural numbers are insufficient for doing subtraction (e.g. 5 minus 7), so we create a bigger set of integers

..., -3, -2, -1, 0, 1, 2, 3, ...

• Integers are insufficient for doing division (e.g. 5 divided by 3) so we create a bigger set of rational numbers

..., -99/7, -10/3, 0, 1/10, 1/5, 3, 6/3, ...

What else is there?

- When we do geometry, other numbers emerge like $\sqrt{2}$.
- However, it satisfies the equation in terms of numbers we already know:
  \[ x^2 - 2 = 0 \]
- The golden ratio satisfies the equation:
  \[ x^2 - x - 1 = 0 \]
- Hence we define the **algebraic numbers**: any number that is the root of an algebraic equation like this.

(The golden ratio)
Algebraic numbers

\[ \sqrt{2}, \quad -\sqrt{5}, \quad \sqrt{7}, \quad \sqrt{11} - \frac{3}{5} \]

Rational numbers

\[ \frac{1}{2}, \quad \frac{5}{7}, \quad -\frac{4}{5} \]

Integers

\[ -3, \quad -5, \quad -\frac{12}{1}, \quad -4/5 \]

Natural numbers

\[ 1, \quad 4, \quad 2, \quad -12 \]

Key question: where’s \( \pi \)?
- Some numbers aren’t even algebraic—they are referred to as transcendental numbers
- In 1882, Lindemann proves that pi is transcendental
Squaring the circle

- Given a rectangle, there is a simple geometrical procedure using a compass and a straight edge to construct a square of equal area.
- What about for a circle? If you could do it, you could geometrically find pi exactly.

Relation to the transcendental nature of $\pi$

- Basic geometry (with a ruler and compass) always gives you algebraic numbers.
- Since $\pi$ is transcendental it will never emerge from a purely geometrical construction, so **squaring the circle is impossible**.
- Many people tried anyway!
The circle-squarers

With the straight ruler I set to make the circle four-cornered.
– Aristophanes, The Birds, 414BC

I have found, by the operation of figures, that this proportion is as 6 to 19. I am asked what evidence I have to prove that the proportion the diameter of a circle has to its circumference is as 6 to 19? I answer, there is no other way to prove that an apple is sour, and why it is so, than by common consent.
– John Davis, The Measure of the Circle, 1854

It is utterly impossible for one to accomplish the work in a physical way; it must be done metaphysically and geometrically, not mathematically.
– A. S. Raleigh, Occult Geometry, 1932
Circle squaromania: Carl Theodore Heisel

- In his 1931 book, he squares the circle, rejects decimal notation, and disproves the Pythagorean theorem
- Finds $\pi = \frac{256}{81}$, and verifies it by checking for circles with radius 1, 2, ..., 9 “thereby furnishing incontrovertible evidence of the exact truth”
- There’s a copy in the Harvard library

Title page from Heisel’s book
Edwin J. Goodwin, Indiana

- “Squares the circle” in 1888; introduced to Indiana house in 1897
- “A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the state of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature of 1897”
- Passed unanimously by the house 67–0, without fully understanding the content of the bill
Edwin J. Goodwin, Indiana

- “It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side”
- Six different values of $\pi$!
- Prof C. A. Waldo (Purdue) visiting at the time, and is shocked that the bill passed
- Persuades Senate to indefinitely postpone action on it
Ramanujan (1887–1920)

- Born to a family without wealth in Southern India
- Math prodigy, but failed college entrance exams
- Went to Cambridge to work with G. H. Hardy in 1913
- Work formed the basis of many modern $\pi$ formulae
- Became chronically ill during WWI
- Returned to India in 1919; died a year later

\[
\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \times \frac{1103 + 26390n}{396^n}
\]
Progress

Doubles approximately every 95 years
Machine calculators

- 1947: Ferguson uses mechanical calculator to compute 710 digits (and finds the error in Shanks’ calculation)
- 1949: ENIAC calculates 2037 digits in 70 hours using the Machin formula – the first automatic calculation of \( \pi \)

ENIAC statistics: 10 feet tall, 1800 ft\(^2\) floor area, 30 tons, 18000 vacuum tubes, 10000 capacitors, 5000 operations a second
Shanks and Wrench: 100,000 decimals (1961)

- Used arctangent formula

\[
\frac{\pi}{4} = 6 \tan^{-1} \frac{1}{8} + 2 \tan^{-1} \frac{1}{37} + \tan^{-1} \frac{1}{239}
\]

- Calculate two terms at a time

\[
\tan^{-1} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{m[(4k+3)m^2 - (4k+1)]}{(16k^2 + 16k + 3)m^4(k+1)}
\]

- Calculated in 8h43m and verified with a second arctangent formula

(This looks complicated, but it's actually just Gregory’s arctangent formula written in a way that’s better suited for computation. It requires fewer divisions, which are relatively slow on a computer.)
Throughout the eighties, the Chudnovsky brothers (USA) and Kanada (Japan) swap the digit record.

Chudnovsky brothers born near Kiev in former USSR.

Both math prodigies and number theorists. Due to Gregory’s illness, they emigrate to the US in 1977.

Why calculate \( \pi \)? Hope that digits provide insight into the number.
Chudnovsky brothers

• Build their own computer called “m zero” in their Upper West Side apartment

• Must face many practical issues: apartment gets to 90° F even with air conditioning

• They derive their own formula

\[
\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}
\]

• Gives 15 digits per term
Computational progress

- Doubles every 28 months
- Doubles every 17 months

Year:
- 1940
- 1960
- 1980
- 2000
- 2020

Number of digits:
- 1
- 10
- 100
- 1000
- 10000
- 100000
- 1000000
- 10000000
- 100000000
- 1000000000
- 10000000000
- 100000000000
Kanada's last record

• Kanada and co-workers computed 1.2 trillion digits in December 2002

• Used Hitachi SR8000 supercomputer with 64 processor nodes, 14.4 billion operations per second per node

• Used two different arctangent formulae

• Takes 601 hours, including the time to move 400 terabytes of data from memory to disk
Last 500 digits

(3) 500 digits ending 1,241,100,000,000-th
(1,241,099,999,501 - 1,241,100,000,000)

| 3716787169 | 6567692125 | 2797286901 | 8503557537 | 6530193499 |
| 3533850167 | 1616469990 | 5984454421 | 7623131551 | 5483436562 |
| 7806800557 | 0748706663 | 5108659327 | 6579461496 | 7987525534 |
| 7689068277 | 7037671632 | 7753867760 | 1776471900 | 9279382597 |
| 6527339324 | 6948904759 | 2872702485 | 4618972965 | 3547547082 |
| 4504016840 | 2350653293 | 6254205392 | 4502959326 | 3809170954 |
| 8310279798 | 7965959470 | 8455199922 | 4435552002 | 5054585883 |
| 0997016164 | 9607402417 | 5296690907 | 5622217705 | 1785600450 |
| 0707455198 | 1744551596 | 6313820124 | 4825046054 | 2311034186 |
| 5591198918 | 2262704528 | 2696896699 | 2856706487 | 3410311045 |
Frequency analysis for the first 1.2 trillion digits


• No known patterns – the digits of pi pass all tests for perfect randomness
• Variation in the digit frequencies is within range of what would be expected for random numbers
Interesting sequences

012345678910 : from 1,198,842,766,717-th of pi
432109876543 : from 149,589,314,822-th of pi
543210987654 : from 197,954,994,289-th of pi
7654321098765 : from 403,076,867,519-th of pi
567890123456 : from 1,046,043,923,886-th of pi
4567890123456 : from 1,156,515,220,577-th of pi
777777777777 : from 368,299,898,266-th of pi
999999999999 : from 897,831,316,556-th of pi
111111111111 : from 1,041,032,609,981-th of pi
888888888888 : from 1,141,385,905,180-th of pi
666666666666 : from 1,221,587,715,177-th of pi
271828182845 : from 1,016,065,419,627-th of pi
314159265358 : from 1,142,905,318,634-th of pi

Search for your phone number in pi at http://www.angio.net/pi/
First 2,133 digits

3.14159265358979323846264338327950288419716939937510582097494459230781640628620
8998628034825342117067982148086513282306647093844609550582231725359408128481117
45028410270193852110555964642629489549303819644288109756659334461284756482337867
8316527120190914564856692346034861045432664821339360726024914127372458700660631
5588174881815209209628292540917153643678925903600113305305458820466521384146951941
5116094330572703657595919530921861173819326117931051185480744623799627495673518
8575272489122793818301194912983367336244065664038602139494639522473719070217986
0943702770539217176293176752384674818467669405132000568127145263560827785771342
757789609173671787214684409012249534301465495853710507922796892589235420199561
1212902196086403441815981362977477130996051870721134999999837297804995105973173
28160963185950244594553469083026425223082533446958035261931188171010003137838752
886587533208381420617177669147303598253490428755456873311595626836823537875937519
5778818778053217122680661300192787661119590921642019893890525720106548586327886
5936153381827968230301952035301852968995773622599413891249721775283479131515574
8572424541506959508295331163617827588590759083817546374649393192550604009277016
7113900984882401285836160356370766010471018194295559619894676783744944825537977
4726847104047534564626028046684259069491293313677028989152104752162056966024058038
150193511253382430033558764024794674326391419927272604269922797682354781636093417
21641219924586315030286182974555706749838505494958858962969596092721079750930295
5321165344987202755960236480665499119881834797753566369807426542527862551818417
5746728909777727938000816470606016145241921732127147723501414149735685481631611
5735255213347574184946843852332390739414334354776241686251898356948556209921922
218427255025425266876717904946016534668049886272327917860857843838797976681454
1009538387386360950680064225125205117392984896084128488626945604241965285022106
6118630674427862203919494504712371378696095636437191728746776465757396241389086
5832645995813390478207590099465764078951269468398352595709825822620522489407726
7194782684826014769909026401363944374553050682034962524517493996514314298091906
Current world records: y-cruncher

- Software developed by Alexander J. Yee and available for download
- Used to set several world records by different individuals
  - August 2010: 5 trillion (Shigeru Kondo)
  - October 2011: 10 trillion (Shigeru Kondo)
  - December 2013: 12.1 trillion (Shigeru Kondo)
  - October 2014: 13.3 trillion (Anonymous, “houkouonchi”)
- Can be attempted with a high-end desktop computer running for several months (but requires many hard drives)

y-cruncher – A multithreaded pi program: http://www.numberworld.org/y-cruncher/
And now for something completely different

• In the 18th century, Buffon considers a striped floor with stripes of a fixed distance apart.

• Consider throwing sticks of same length onto the strips. The chance that they will overlap a stripe is $2/\pi$!

This can also be generalized to case when stripes and sticks are of different lengths.

Georges-Louis Leclerc, comte de Buffon (1707–1788)
Memorizing pi

- A fun party trick
- 1970’s: world record held by Simon Plouffe (4096 digits)
- 1983: Rajan Mahadevan sets record with 31,811 digits
- 1995: Hiroyuki Goto sets record with 42,000 digits
- 2004: Lu Chao sets record with 67,890 digits
How I wish I could remember pi.

3 1 4 1 5 9 2

See, I have a rhyme assisting my feeble brain, its tasks oft-times resisting.

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics.
Pi Mnemonics

Sir, I bear a rhyme excelling
In mystic force and magic spelling
Celestial sprites elucidate
All my own striving can't relate
Or locate they who can cogitate
And so finally terminate. Finis.

3.1415926535897932384626433832795028841971...

Pi mnemonic road block!
Use either 10-letter words or special punctuation to mark a zero.
Star Trek episode #43: The wolf in the fold

In this episode, Kirk and Spock fool a computer by making it compute pi forever

Computer, this is a Class A compulsory directive. Compute to the last digit the value of pi.

– Spock
References


Chris H. Rycroft
chr@seas.harvard.edu
http://seas.harvard.edu/~chr/

Slides will be posted at http://seas.harvard.edu/~chr/present/amherst_pi_day.pdf