1 Example

Exercise. Suppose \( T(1) = 3 \) and \( T(n) = 3T(n/2) + n \). How would you find \( T(8) \)? The point of this exercise is the process.

Solution.
Expand and substitute using the formula for the recurrence:

\[
T(8) = 3T(4) + 8 \\
= 3 \left[3T(2) + 4 \right] + 8 = 9T(2) + 20 \\
= 9 \left[3T(1) + 2 \right] + 20 = 27T(1) + 38 = 119
\]

This is the same approach that’s used to prove the Master Theorem.

2 Master Theorem

Start with a recurrence \( T(n) = aT(n/b) + cn^k \) (supposing that \( T(p_0) = q_0 \) for constants \( p_0 \) and \( q_0 \)) and expand:

\[
T(n) = aT(n/b) + cn^k \\
= a \left[ aT(n/b^2) + c \left( \frac{n}{b} \right)^k \right] + cn^k = a^2T(n/b^2) + cn^k \left( 1 + \frac{a}{b^k} \right) \\
\vdots \\
= a^sT(n/b^s) + cn^k \left[ \left( \frac{a}{b^k} \right)^s + \left( \frac{a}{b^k} \right)^{s-1} + \ldots + \frac{a}{b^k} + 1 \right]
\]

We stop expanding when we reach the base case, when \( \frac{n}{b^s} = p_0 \). This occurs after \( s \approx \log_b \left( \frac{n}{p_0} \right) = \log_b n + \) constant iterations. Notice that the expression is split into two terms. The asymptotic form of \( T(n) \) is just a competition between these two terms to see which one dominates.

The second term has a geometric sum: using the formula for a geometric sum gives:

\[
T(n) = a^s q_0 + cn^k \left[ \frac{1 - \left( \frac{a}{b^k} \right)^{s+1}}{1 - \frac{a}{b^k}} \right]
\]

Exercise. Use the above expansion to derive the case of the Master Theorem for \( a < b^k \).
Solution. Here, \( \frac{a}{b^k} < 1 \), and as \( n \) (and therefore \( s \)) grows large the sum of the above geometric series is dominated by the constant term \( \frac{1}{\frac{a}{b^k}} = \Theta(1) \). So \( T(n) = \Theta(a^s) + \Theta(n^k) \). Using our expression for \( s \):

\[
a^s = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a}) = o(n^k)
\]

since \( a < b^k \) means that \( \log_b a < k \). We therefore get that \( T(n) = o(n^k) + \Theta(n^k) = \Theta(n^k) \).

Exercise. Now derive the Master Theorem for \( a > b^k \).

Solution. Proceeding like the previous case, the geometric sum is now dominated by the:

\[
\left( \frac{\frac{a}{b^k}}{\frac{a}{b^k}} \right)^{s+1} = \Theta \left( \left( \frac{a}{b^k} \right)^s \right)
\]
term. Then the second term of \( T(n) \) is:

\[
cn^k \cdot \Theta \left( \left( \frac{a}{b^k} \right)^{\log_b n} \right) = cn^k \cdot \Theta \left( \frac{n^{\log_b a}}{n^k} \right) = \Theta \left( n^{\log_b a} \right)
\]

This along with the result from the previous exercise that \( a^s = \Theta \left( n^{\log_b a} \right) \) gives that \( T(n) = \Theta \left( n^{\log_b a} \right) \).

Exercise. Derive the Master Theorem for \( a = b^k \).

Solution. Every term in the geometric series is now 1. There are \( s + 1 \) terms, so the second term of \( T(n) \) becomes:

\[
cn^k(s + 1) = \Theta \left( n^{k \log_b n} \right) = \Theta \left( n^k \log n \right)
\]

The first term of \( T(n) \) is \( \Theta \left( n^{\log_b a} \right) = \Theta \left( n^k \right) \) so the second term dominates and \( T(n) = \Theta \left( n^k \log n \right) \).

Qualitatively, if \( a > b^k \), the bottleneck of the recurrence is the number of recursive calls we have to make. Otherwise, it’s the extra work done during each call (i.e. the \( cn^k \) term) that dominates the runtime.