Today

PCP & Inapproximability

"Inapproximability Result"

Thm: [Hastad '97]:

$\text{MAX 3 SAT}$ is hard to approximate to within $7 + \epsilon$.

\[ \uparrow \]

Lemma: \exists polytime reduction $T$ s.t.

$G$ 3-colorable $\Rightarrow T(G) = C_1, C_2, \ldots, C_m$

all $C_i$ satisfiable

$G$ not 3-colorable $\Rightarrow T(G) = C_1, \ldots, C_m$

any assignment s.t.

\[ \leq \frac{7}{8} + \epsilon \]
$T = \text{"gap-creating reduction"}$

Most inapprox. Results use "gap-preserving" reduction

\[ \text{3SAT} \xrightarrow{\text{gap}} \text{PCP} \xrightarrow{\text{gap preserving}} \text{3SAT} \xrightarrow{\text{gap preserving}} \text{Clique} \]

Gap-Preserving Reduction (by example)

$\text{3SAT} \xrightarrow{\text{m clauses}} \text{Clique} \xrightarrow{\text{7m vertices}}$

$\text{OPT} = t$

\[ \text{max \#sat. clauses} \]

clique size = $t$
**Claim:**

Assignment $\Rightarrow$ Chique

Chique $\Rightarrow$ Partial Assignment

Chique has $\leq$ one vertex from each row

$\Rightarrow \# \text{ rows} = \# \text{ satisfying clauses}$
PCP = ?

Defined for language L by verifier V

V: - prob. algorithm;
   - reads "Theorem" fully
   - queries "Proof" briefly
   - Accepts valid proofs
   - Rejects invalid theorems w.h.p.

E.g. PCP for graph coloring

\[ G \text{ 3 colorable } \Rightarrow \exists \text{ Proof accepted w.h.p.} \]
\[ G \text{ not 3 colorable } \Rightarrow \forall \text{ Proof, accepted w. low prob.} \]
[Alsted '97]: \exists \text{ verifier } V \text{ running in polytime, makes 3 queries to proof,}
accepts valid proofs w.p. $\approx 0.999 - \frac{1}{2}$
rejects invalid claims w.p. $\approx 0.999 + \frac{1}{2}$

- Furthermore, every "check" of the form
  "is $\Pi_i \oplus \Pi_j \oplus \Pi_k = 0/1$?"
- Proof is of poly size $S = n^{o(1)}$

PCP \implies \text{Max 3SAT}

One variable for every bit of proof.

One "constraint" for every check
  $= 4$ clauses for every check

Example

\begin{align*}
\Pi_i \oplus \Pi_j \oplus \Pi_k = 0 & \implies \land (x_i \lor x_j \lor \overline{x_k}) \\
\land (\overline{x_i} \lor x_j \lor x_k) \\
\land (\overline{x_i} \lor \overline{x_j} \lor x_k)
\end{align*}
Analysis

1. If $G$ is $3$-colorable $\Rightarrow$ $\exists$ proof s.t. $V$ accepts $w.p. \approx 1$

2. $\Rightarrow \exists$ assignment $a_1, \ldots, a_s$ to $X_1, \ldots, X_s$

   set almost all clauses

3. If not $3$-colorable $\Rightarrow$ $\forall$ proofs $V$ accepts $w.p. \approx \frac{1}{2}$

   $\Rightarrow \forall$ assignments $a_1, \ldots, a_s$

   for about $\frac{1}{2}$ the checks

   only $3/4$ clauses set.

   $\Rightarrow$ any assignment $s.t.$ at most $\frac{7}{8}$ falsify $\phi$ clause
PCP for graph coloring = Proof?

Recall: Cook-Levin Thm, NP-completeness

+ Short Proof $\in$ NP!

$\implies$ given any "theorem" $T$, system of logic $L$ & length $n$ of proof

size

$\exists$

Can construct in time $\text{poly}(n)$, graph $G = G(T, L, n)$:

$G$ is 3-colorable ($\iff$ $T$ has a proof of length $\leq n$)

Moral: 3-coloring PCP witnesses
PCP Constructions

(too long to fit this margin)

Main Idea: "Speak in Polynomials"

\[ A(1) \ldots A(n^2) \], where

\[ A(x) \text{ has degree } \leq N \]

\[ \forall i \in \{1 \ldots n^2\} \]

\[ A(i) = a_i \]

• Graph 3 coloring "logically"

given \[ E : [n] \times [n] \rightarrow \{0,1\} \]

\[ \exists X : [n] \rightarrow \{0,1,2\} \]

st. \[ \forall (i,j) \ E(i,j) = 0 \rightarrow X(i) \neq X(j) \]
Graph 3-coloring "algebraically"

Given: \( E(y,z) \) poly of deg \( \leq n \)

\[ \exists X(z) \]

s.t. \( \forall z \in [n] \)

\[ X(z) \cdot (X(z) - 1) \cdot (X(z) - 2) = 0 \] ?

\[ \exists y, z \in [n] \]

\[ E(y,z) \cdot \text{TT} \ (X(y) - X(z) - 1) = 0 \]
\( \lambda \in \{-2,-1,1,2\} \)

"Polynomial Relation" - "Easy to check"

(possibly)

-- Details Omitted --
Approx. today

Strong Optimization (extension of LP)

Strong "PCP" techniques suggest nearly optimal analysis of many many many optimization problems

modulo one conjecture

is the following problem hard?

\[
\begin{cases}
X_i + X_j = a_{ij} \pmod{p} \\
\text{for } (i,j) \in E
\end{cases}
\]

Can you distinguish most being satisfiable from most being unsatisfiable?

"Unique games conjecture": NO!