IV. **OPTICAL RESONATORS:**

**STABILITY CRITERIA FOR RESONATORS AND PERIODIC OPTICAL STRUCTURES BY RAY OPTIC ANALYSIS**

Consider a prototypical periodic guiding lens system or an equivalent resonator.

Using the appropriate ABCD matrix with the indicated reference planes, we may write

\[ \rho_{m+1} = A \rho_m + B \rho_m' \]  

[ IV-1a ]

and

\[ \rho_{m+1}' = C \rho_m + D \rho_m' \]  

[ IV-1b ]

Where for reference, we see that

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-d/f_2 & -d/f_1 + d^2/f_2 & 2d-d^2/f_2 \\ d/f_1f_2 & -1/f_1 - 1/f_2 & 1-d/f_2 \end{bmatrix}
\]
From the first equation we write

\[ \rho_m' = \frac{\rho_{m+1} - A \rho_m}{B} \quad \text{and} \quad \rho_{m+1}' = \frac{A \rho_m}{B}. \]

and substitute into Equation \[ IV-1b \] to obtain

\[ \rho_{m+2} - [A + D] \rho_{m+1} + [AD - BC] \rho_{m+1} = 0 \quad \text{[ IV-2a ]} \]

The determinant of the coefficients \( [A \, D - BC] = 1 \) so that

\[ \rho_{m+2} - [A + D] \rho_{m+1} + \rho_{m+1} = 0. \quad \text{[ IV-2a ]} \]

We see that

\[ \frac{A + D}{2} = \beta = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2 f_1 f_2} = -1 + 2 \left( 1 - \frac{d}{2 f_1} \right) \left( 1 - \frac{d}{2 f_2} \right) \quad \text{[ IV-3] } \]

Thus, stable ray propagation may characterized by **bound solutions** of the form \( \rho_m = \rho_0 \exp(i \, m \phi) \) which are possible if and only if

\[ \exp(i \phi) + \exp(-i \phi) = 2 \cos \phi = A + D = 2 \beta \quad \text{or} \quad \cos \phi = \beta \quad \text{[ IV-4] } \]

Therefore propagation is stable -- *i.e.* the rays are confined -- when \( |\beta| \leq 1 \) so that

\[ 0 \leq \left( 1 - \frac{d}{2 f_1} \right) \left( 1 - \frac{d}{2 f_2} \right) \leq 1. \quad \text{[ IV-5] } \]

Ray stability of rays in a periodic system may be usefully characterized in terms of the variables \( u_1 = 1 - \frac{d}{2 f_1} \) and \( u_2 = 1 - \frac{d}{2 f_2} \) as follows:
Stability (Confinement) Diagram for Periodic Systems
Stability of a Spherical Mirror Resonators -- Using Solutions of the Paraxial Wave Equation:

Consider a Hermite-Gaussian mode confined in an asymmetrical spherical cavity:

In order to sustain a resonant mode in such a cavity, the beam's radius of curvature must match each mirror's radius of curvature at the mirror's surface and, thus, the following conditions must hold (see Equation [III-20]):

\[ R_1 = z_1 + \frac{L_F^2}{z_1} \quad \text{and} \quad R_2 = -z_2 - \frac{L_F^2}{z_2} \quad \text{[IV-5a]} \]

where \( d = z_1 + z_2 \). Hence, we see that

\[ z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4L_F^2}}{2} \quad \text{and} \quad z_2 = \frac{-R_2 \pm \sqrt{R_2^2 - 4L_F^2}}{2} \quad \text{[IV-5b]} \]
with a lot of algebra we can show that

\[
L_f^2 = \left( \frac{\pi w^2(0)}{\lambda} \right)^2 = -d u_1 u_2 R_1 R_2 \left[ d + u_1 R_1 - u_2 R_2 \right] \frac{1}{u_1 R_1 - u_2 R_2} \]

[ IV-6 ]

where now \( u_1 = 1 - d/R_1 \) and \( u_2 = 1 + d/R_2 \).

For a symmetric resonator \( R_2 = -R_1 \) and \( u_1 = u_2 \)

\[
L_f^2 = \left( \frac{\pi w^2(0)}{\lambda} \right)^2 = \frac{d}{4} \left( 2R - d \right) \]

[ IV-7a ]

and

\[
w(z) = w(-z) = \left[ \frac{d\lambda}{2\pi} \right]^\frac{\gamma}{\pi} \left[ \frac{2R^2}{d(R-d/2)} \right]^\frac{\pi}{4} \]

[ IV-7b ]

For an asymmetric resonator, it can be shown with a bit more algebra that

\[
w(z) = \left[ \frac{d\lambda}{\pi} \right]^\frac{\gamma}{\pi} \left[ \frac{u_1}{u_1(1-u_1 u_2)} \right]^\frac{\pi}{4} \]

[ IV-8a ]

\[
w(-z) = \left[ \frac{d\lambda}{\pi} \right]^\frac{\gamma}{\pi} \left[ \frac{u_2}{u_2(1-u_1 u_2)} \right]^\frac{\pi}{4} \]

[ IV-8b ]

As a measure of the effect of resonator length and mirror radius on diffraction loss consider the ratio:

\[
\frac{w(z)}{w_{\text{conf}}(z)} = \frac{w(-z)}{w_{\text{conf}}(-z)} = \left[ \frac{d}{R(1 - d/2R)} \right]^\frac{\pi}{4} \]

[ IV-9 ]
where \( w_{\text{conf}}(z_1) = \sqrt{2} w(0) \) is beam width at the mirror for the confocal configuration — i.e., when both mirrors have their focal points at the mid-point of the cavity.

Resonance Frequencies of the Optical Resonator:
From Equation [ III-28 ], we see the the “round-trip” cold resonance condition for a Hermite-Gaussian mode is given by

\[
k d - (n + m + 1) \left[ \tan^{-1} \left( \frac{z_1}{L_f} \right) + \tan^{-1} \left( \frac{z_2}{L_f} \right) \right] = N\pi \quad \text{[ IV-10a ]}
\]

where \( N \) is an integer. In terms of frequency, the resonance condition is
\[ \omega = \frac{c}{d} \left\{ N\pi + (n + m + 1) \left[ \tan^{-1} \left( \frac{z_1}{L_T} \right) + \tan^{-1} \left( \frac{z_2}{L_T} \right) \right] \right\} \quad [\text{IV-10b}] \]

After much algebra, it can be shown that:

\[ \omega = \frac{c}{d} \left[ N\pi + (n + m + 1) \cos \sqrt{u_1 u_2} \right] \quad [\text{IV-10b}] \]