## V.   ELEMENTARY LASER MODELS -- RATE EQUATION APPROACH:

POPULATION INVERSION AND "LASERING" IN A THREE-LEVEL SYSTEM

Obviously, the trick in building masers and lasers has been to find means to "invert" the equilibrium population of states.[22]   First conceived by Charles Townes in 1951, Gordon, Zeiger and Townes in 1954 demonstrated the first ammonia beam maser.   Degenerate quantum states of the ammonia molecule (NH_3) are split by the tunneling of the nitrogen atom across the plane formed by the three hydrogen atoms.   In the Columbia maser, thermal populations of molecules in the split ground state (23,800 MHz) are spatially separated by an electrostatic field to achieve population inversion in the maser resonant cavity. While this two-level system was clearly seminal, it was the multi-level paradigm of
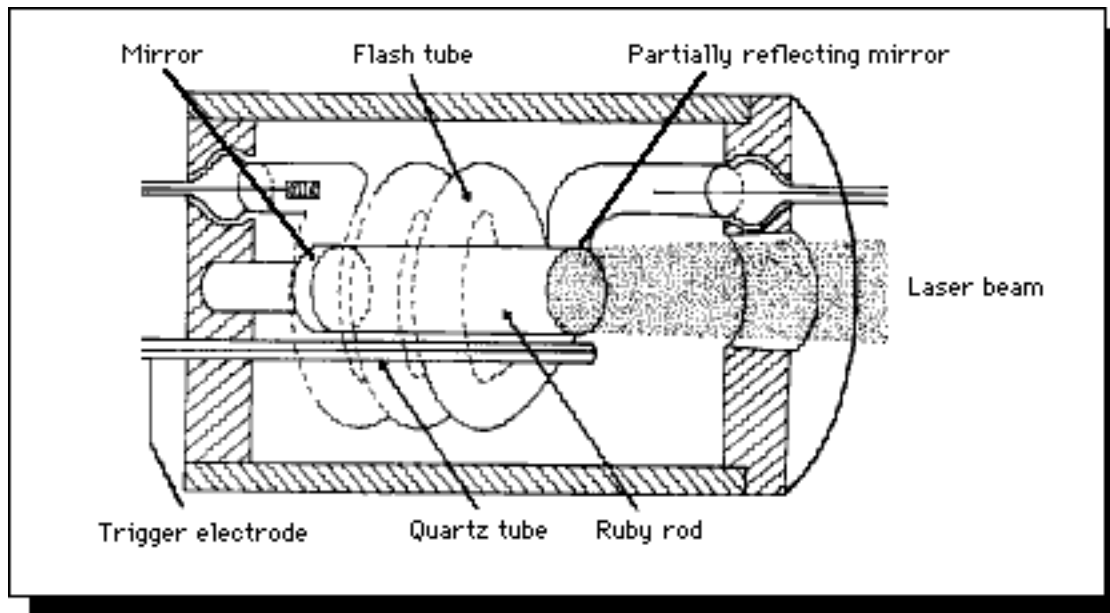
THE CONFIGURATION OF THEODORE MAIMAN'S RUBY LASER (1960)

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[22]   Recently the subject of lasing has become of considerable interest. See, for Thomas Mossberg's very
interesting Web Seminar on the subject (http://decryptor.uoregon.edu/~mosswww/moss-seminar.html).

Bloembergen's three-level ruby maser which galvanized the search for possible laser systems.

To quote from *A Laser Tutorial* (http://members.aol.com/WSRNet/tut/t1.htm)

"Finding substances in which a population inversion can be set up is central to the development of new kinds of lasers. The first material used was synthetic ruby. Ruby is crystalline alumina (Al_2O_3) in which a small fraction of the Al^{3+} ions have been replaced by chromium ions, Cr^{3+}. It is the chromium ions that give rise to the characteristic pink or red colour of ruby and it is in these ions that a population inversion is set up in a ruby laser. In a ruby laser, a rod of ruby is irradiated with the intense flash of light from xenon-filled flashtubes. Light in the green and blue regions of the spectrum is absorbed by chromium ions, raising the energy of electrons of the ions from the ground state level to one of the broad F bands of levels. Electrons in the F bands rapidly undergo non-radiative transitions to the two metastable E levels. A non-radiative transition does not result in the emission of light; the energy released in the transition is dissipated as heat in the ruby crystal. The metastable levels are unusual in that they have a relatively long lifetime of about 4 milliseconds (4×10^{-3} s), the major decay process being a transition from the lower level to the ground state. This long lifetime allows a high proportion (more than a half) of the chromium ions to build up in the metastable levels so that a population inversion is set up between these levels and the ground state level. This population inversion is the condition required for stimulated emission to overcome absorption and so give rise to the amplification of light. In an assembly of chromium ions in which a population inversion has been set up, some will decay spontaneously to the ground state level emitting red light of wavelength 694.3 nm in the process. This light can then interact with other chromium ions that are in the metastable levels causing them to emit light of the same wavelength by stimulated emission. As each stimulating photon leads to the emission of two photons, the intensity of the light emitted will build up quickly. This cascade process in which photons emitted from excited chromium ions cause stimulated emission from other excited ions…"

"The ruby laser is often referred to as an example of a three-level system. More than three energy levels are actually involved but they can be put into three categories. These are; the lower level form which pumping takes place, the F levels into which the chromium ions are pumped, and the metastable levels from which stimulated emission occurs. Other types of laser operate on a four level system and , in general, the mechanism of amplification differs for different lasing materials. However, in all cases, it is necessary to set up a population inversion so that stimulated emission occurs more often than absorption."

The following rate equation analysis demonstrates the critical conditions necessary to obtain sufficient inversion:

A "PUMPED" THREE-LEVEL QUANTUM SYSTEM
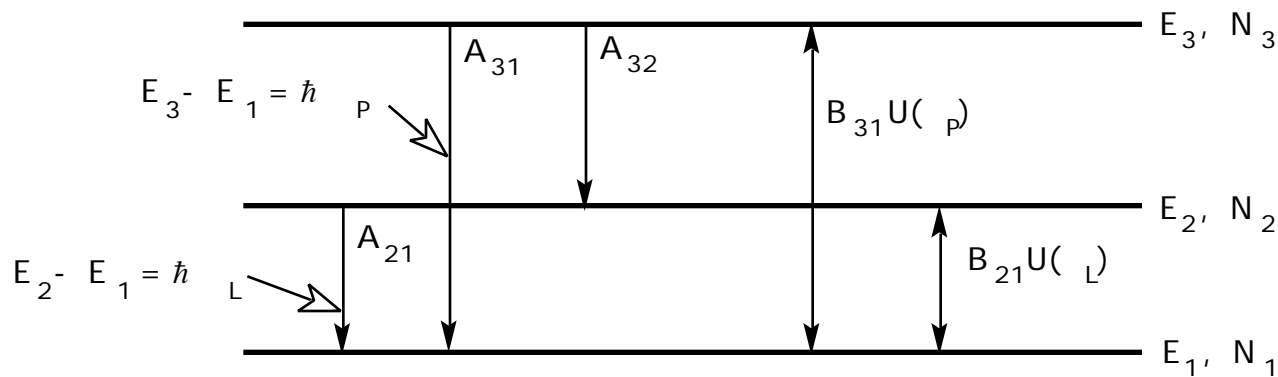
Master rate equations which include "laser" and "pumping" radiation, but in which thermal radiation is neglected:

$$\frac{d}{dt} N_1 = N_3\, A_{31} + N_2\, A_{21} + \left(N_3 - N_1\right) B_{31}\, U\!\left(\nu_P\right) + \left(N_2 - N_1\right) B_{21}\, U\!\left(\nu_L\right) \quad [\text{ V-1a }]$$

$$\frac{d}{dt} N_2 = N_3\, A_{32} - N_2\, A_{21} - \left(N_2 - N_1\right) B_{21}\, U\!\left(\nu_L\right) \qquad\qquad [\text{ V-1b }]$$

$$\frac{d}{dt} N_3 = -N_3 A_{31} - N_3 A_{32} - (N_3 - N_1) B_{31} U(\ _P) \qquad \text{[V-1c]}$$

Of course, only two of these equations are independent, since we assume that there are a fixed number of systems N distributed over these energy levels. In what follows, we define

$$R = U(\ _P) B_{31} (N_1 - N_3)/N$$

as the ***pumping rate*** with $N_3 = N - N_1 - N_2$.

We now study the steady state solutions of two independent rate equations -- *i.e.*

$$0 = N_3 A_{32} - N_2 A_{21} - (N_2 - N_1) B_{21} U(\ _L)$$
$$0 = -N_3 (A_{31} + A_{32}) - (N_3 - N_1) B_{31} U(\ _P) \ - N_3 (A_{31} + A_{32}) + R N \qquad \text{[V-2a]}$$

which may be written

$$N_1 [A_{32} - B_{21} U(\ _L)] + N_2 [A_{32} + A_{21} + B_{21} U(\ _L)] = N A_{32}$$
$$N_1 [A_{31} + A_{32}] + N_2 [A_{31} + A_{32}] = N (A_{31} + A_{32}) - R N \qquad \text{[V-2b]}$$

By messy, but trivial algebra, we find

$$N_1 = -\frac{N[R (A_{32} + A_{21} + B_{21} U(\ _L)) - (A_{31} + A_{32})(A_{21} + B_{21} U(\ _L))]}{[A_{31} + A_{32}][A_{21} + 2B_{21} U(\ _L)]}$$

$$N_2 = \frac{N[R (A_{32} - B_{21} U(\ _L)) + (A_{31} + A_{32}) B_{21} U(\ _L)]}{[A_{31} + A_{32}][A_{21} + 2B_{21} U(\ _L)]} \qquad \text{[V-3]}$$

Therefore, the all important population difference may be expressed as

$$N_2 - N_1 = \frac{N\left[R\left(2A_{32} + A_{21}\right) - A_{21}\left(A_{31} + A_{32}\right)\right]}{\left[A_{31} + A_{32}\right]\left[A_{21} + 2B_{21} U(\nu_L)\right]} \qquad [\text{ V-4a }]$$

or, with regrouping of terms, as

$$N_2 - N_1 = \frac{N\left[R\left(2A_{32} + A_{21}\right) - A_{21}\left(A_{31} + A_{32}\right)\right]}{A_{21}\left[A_{31} + A_{32}\right]\left(1 + \dfrac{2B_{21} U(\nu_L)}{A_{21}}\right)}$$

$$= \frac{N\left[R\left(2A_{32} + A_{21}\right) - A_{21}\left(A_{31} + A_{32}\right)\right]}{A_{21}\left[A_{31} + A_{32}\right]\left(1 + \dfrac{I(z, \nu_L)}{I_{crit}}\right)} \qquad [\text{ V-4 b}]$$

where $I_{crit} = (c/n)\left(A_{21}/2B_{21}\right)$. Since the spatial change in the Poynting vector associated with a beam propagating through the laser medium is equal to the rate of energy density lost (or gained) by the beam, **you can show** that

$$\frac{1}{I(z,\nu)}\frac{d}{dz} I(z,\nu) = -\left(N_1 - N_2\right)(c/n)\left(\hbar\nu B_{21}\right)\left(F(\nu)/V\right) \qquad [\text{ V-5 }]$$

where $F(\nu)$ is the *line shape* function of the $E_2 - E_1$ absorption line. Using Equation [V-4b], we may write

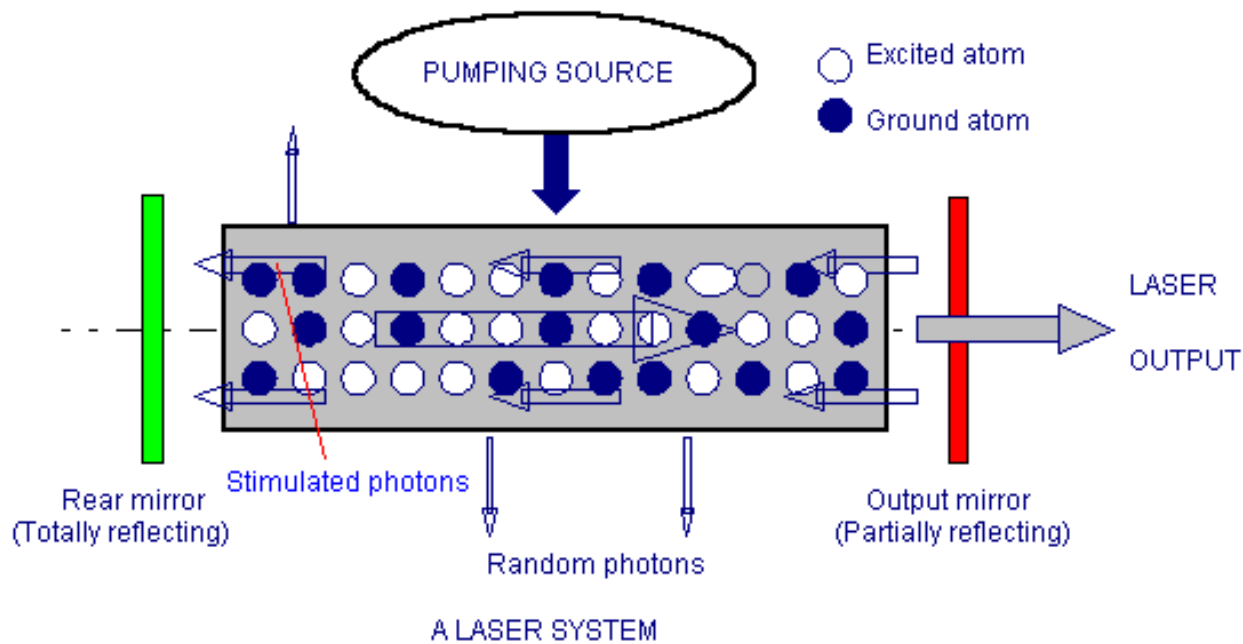$$\frac{1}{I(z,\nu)}\frac{d}{dz} I(z,\nu) = \frac{R\left(2A_{32} + A_{21}\right) - A_{21}\left(A_{31} + A_{32}\right)}{A_{21}\left[A_{31} + A_{32}\right]\left(1 + \dfrac{I(z,\nu)}{I_{crit}}\right)}\; \frac{c}{n}\; \frac{N\hbar\nu B_{21} F(\nu)}{V} \qquad [\text{ V-6 }]$$

Therefore, we may write

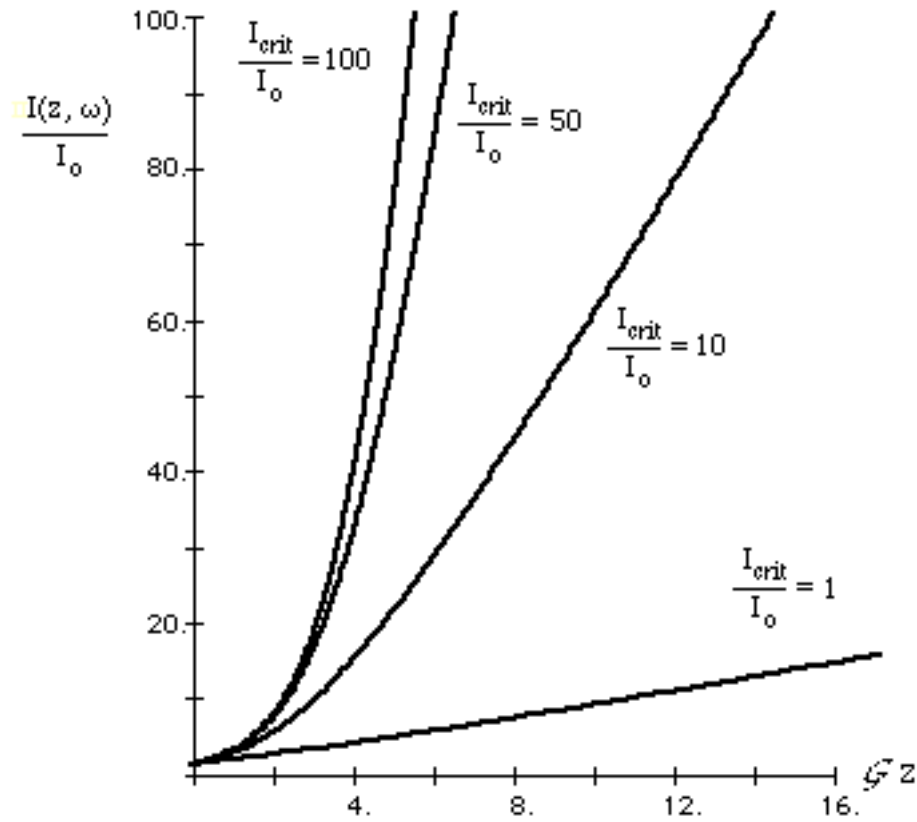$$\frac{1}{I(z,\ )}\left[1+\frac{I(z,\ )}{I_{crit}}\right]\frac{d}{dz}I(z,\ )=\mathcal{G} \qquad [\text{ V-7a }]$$

where we identify the *gain of the medium* as

$$\mathcal{G}=\frac{R\left(2A_{32}+A_{21}\right)-A_{21}\left(A_{31}+A_{32}\right)}{A_{21}\left[A_{31}+A_{32}\right]}\ \frac{c}{n}\ \frac{N\,\hbar\ \ B_{21}\,F(\ )}{V} \qquad [\text{ V-7b }]$$

A LASER SYSTEM

Thus, we have a interesting and useful model for the "gain behavior" (growth in intensity with position) of a beam as a function of initial intensity -- *viz*.

$$\ln\frac{I(z,\ )}{I_o}+\frac{I(z,\ )-I_o}{I_{crit}}=\mathcal{G}\,z \qquad [\text{ V-8 }]$$

$\dfrac{I(z, \omega)}{I_o}$

$\dfrac{I_{crit}}{I_o} = 100$

$\dfrac{I_{crit}}{I_o} = 50$

$\dfrac{I_{crit}}{I_o} = 10$

$\dfrac{I_{crit}}{I_o} = 1$

$\mathcal{G} z$

For $I_o << I_{crit}$ $I(z, \quad) = I_o \exp(\mathcal{G} z)$ "Inverse Beer's Law"

For $I_o >> I_{crit}$ $I(z, \quad) = I_o + I_{crit} \, \mathcal{G} z$ Linear regime

Now consider elementary laser operation with feedback provided by two mirrors separated by a distance L. We first calculate the famous formula for ***threshold gain*** from the following consistency condition:

$$I_o \exp\left(- \; 2L\right)\exp\left(\mathcal{G}_{th} \; 2L\right) r_1 \, r_2 = I_o \qquad \text{[V-9a]}$$

or

$$\boxed{\mathcal{G}_{th} \; = \; + \frac{1}{2L} \ln \frac{1}{r_1 \, r_2}} \qquad \text{[V-9b]}$$

where   is included to account for residual cavity losses and the r_i 's are the respective reflectivity's of the two mirrors. At high power -- *i.e.* at powers well above threshold -- the consistency condition becomes (neglecting residual absorption)

$$I_o \quad r_1 \, r_2 \left[I_o + I_{crit} \, \mathcal{G} \, 2L \right] \qquad \text{[V-10a]}$$

or

$$I_o \quad 2I_{crit} \, \mathcal{G} \, L \; \left[\frac{1}{r_1 \, r_2} -1\right]^{-1} \qquad \text{[V-10b]}$$

From Equation [V-7b] we may write

$$\frac{\mathcal{G}}{\mathcal{G}_{th}} \; = \frac{R - R_o}{R_{th} - R_o} \qquad \text{[V-11a]}$$

where $\qquad R_o = \dfrac{A_{21}\left(A_{31} + A_{32}\right)}{\left(2A_{32} + A_{21}\right)}$ and $\qquad R_{th} - R_o = \mathcal{G}_{th} \, \dfrac{A_{21}\left[A_{31} + A_{32}\right]}{\left(2A_{32} + A_{21}\right)} \; \dfrac{n}{c} \; \dfrac{V}{N \, \hbar \quad B_{21} \, F(\;)}$ [V-11b]

Therefore Equation [V-10b] becomes

$$I_o(R) \quad \frac{2I_{crit} \, \mathcal{G}_{th} \, L}{\dfrac{1}{r_1 \, r_2} -1} \; \frac{R - R_o}{R_{th} - R_o} \; = \; \frac{2I_{crit} \, \mathcal{G}_{th} \, L}{\left[\dfrac{1}{r_1 \, r_2} -1\right]\left[R_{th} - R_o\right]} \; \left(R - R_o\right) = C \left(R - R_o\right) . \qquad \text{[V-12]}$$

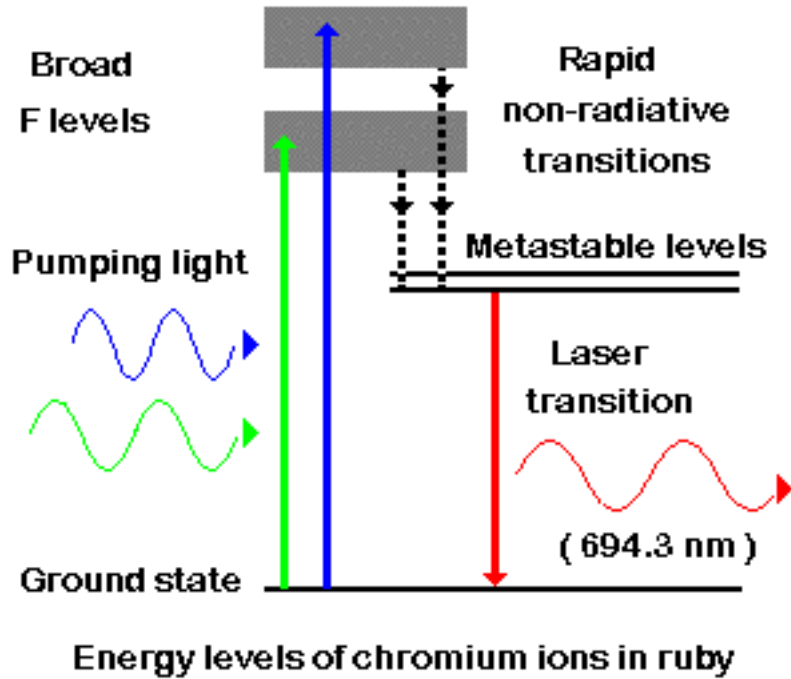This result, which is plotted below, is archetypal of the threshold behavior all lasers.

LASER LIGHT OUTPUT "WELL ABOVE" THRESHOLD
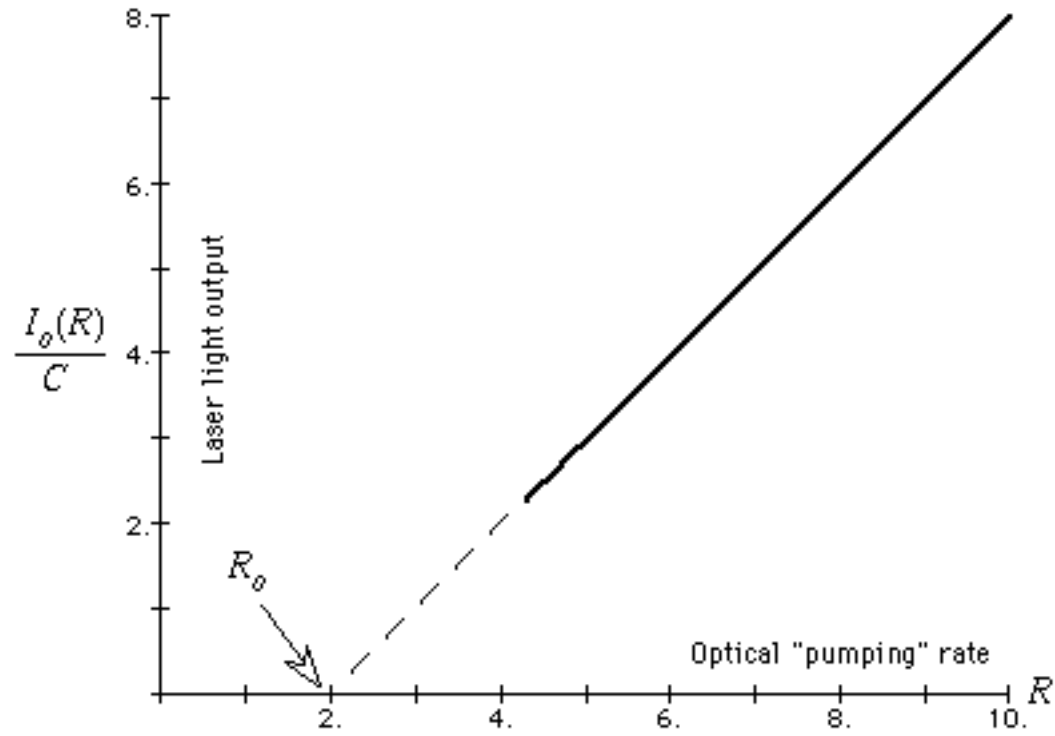
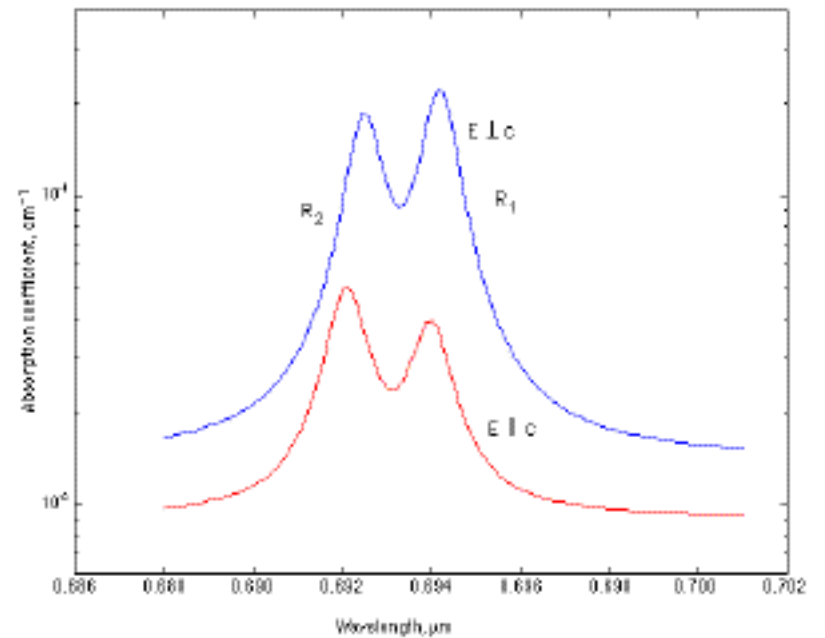## Appendix: Common Lasers

Pink Ruby Laser Laser: the first pulsed laser

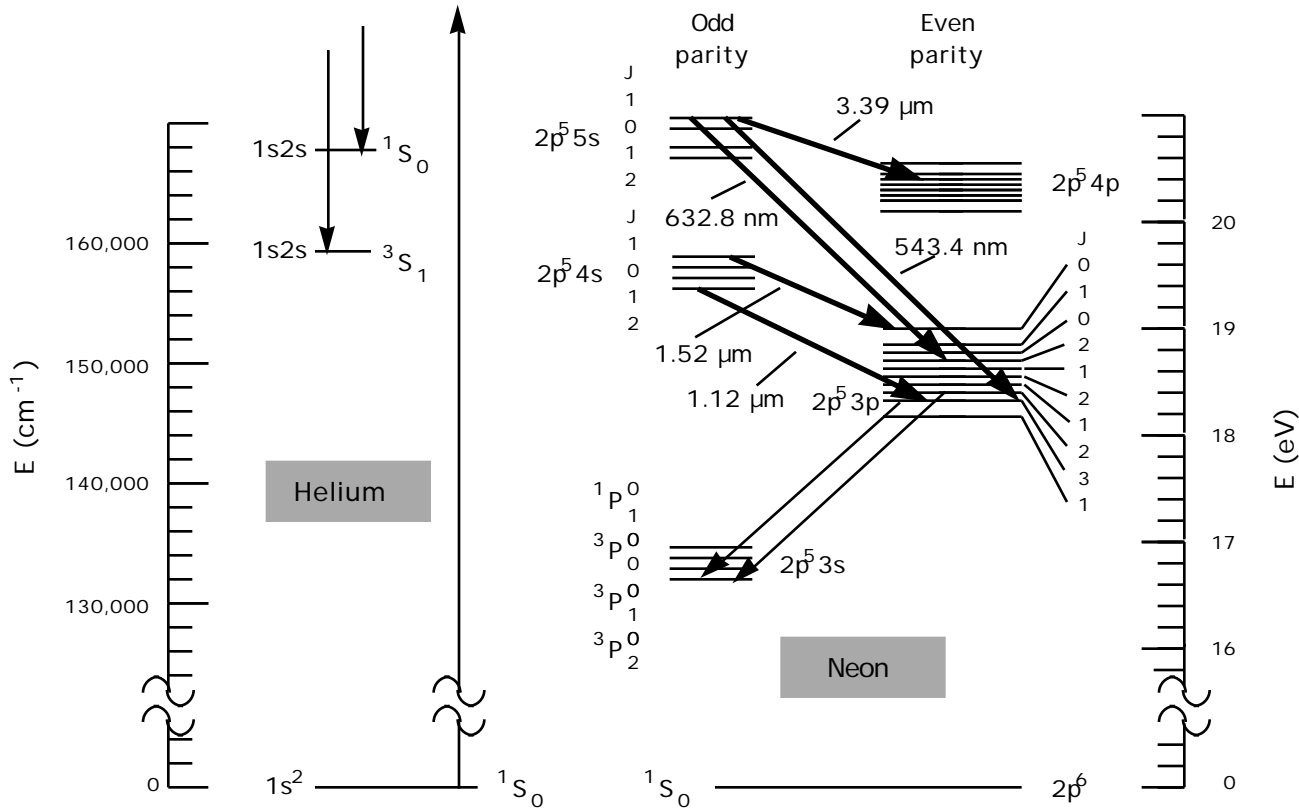Energy Levels of Cr^{3+} in a Trigonal Crystal Field   Absorption Spectrum of R1 and R2 of Cr^{3+} in Ruby

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Absorption Spectrum of R1 and R2 of Cr^{3+} in Rub

HeNe Neutral Gas Laser: the first continuous laser

Mechanism: A dc or rf electrical discharge is established in a gas mixture of helium (90%) and neon (10%). The helium atoms are excited to long-lived metastable states by a variety of processes involving energetic electron impact. The excitation energy of the helium metastables is transfered to neon atoms by resonance scattering to yield **laser-active** neon metastables.

Particularly strong emissions

$1s^2 2s^2 2p^5 5s: {}^1P_1$	$1s^2 2s^2 2p^5 3p: {}^3S_1$	$_{vacuo} = 0.5435$ µm;	$_{air} = 0.5433$ µm
$1s^2 2s^2 2p^5 5s: {}^1P_1$	$1s^2 2s^2 2p^5 3p: {}^3P_2$	$_{vacuo} = 0.6330$ µm;	$_{air} = 0.6328$ µm

$1s^2 2s^2 2p^5 4s: {}^1P_1$ $1s^2 2s^2 2p^5 3p: {}^3P_2$ vacuo = 1.1526 μm; air = 1.1523 μm

$1s^2 2s^2 2p^5 4s: {}^1P_1$ $1s^2 2s^2 2p^5 3p: {}^1S_0$ vacuo = 1.5235 μm; air = 1.5231 μm

$1s^2 2s^2 2p^5 5s: {}^1P_1$ $1s^2 2s^2 2p^5 4p: {}^3P_2$ vacuo = 3.3922 μm; air = 3.3913 μm

Rare Earth Ion Solid State Lasers:

Neutral atom electronic configurations:

noble gas core xeon: [Xe] $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 5s^2 5p^6$

rare earth atoms: $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10} 4f^n 5s^2 5p^6 6s^2 = $ [Xe] $4f^n 6s^2$

Trivalent ion electronic configurations of common laserable rare earth ions:

|		
praseodymium (Pr^{+3}):	= [Xe] $4f^2$	Ground state: $^3H_4$
neodymium (Nd^{+3}):	= [Xe] $4f^3$	Ground state: $^4I_{9/2}$
europium (Eu^{+3}):	= [Xe] $4f^6$	Ground state:
holmium (Ho^{+3}):	= [Xe] $4f^{10}$	Ground state:
erbium (Er^{+3}):	= [Xe] $4f^{11}$	Ground state: $^4I_{15/2}$
thulium (Tm^{+3}):	= [Xe] $4f^{12}$	Ground state: $^3H_6$
ytterbium (Yb^{+3}):	= [Xe] $4f^{13}$	

Important characteristic of rare earth ions: Outer 5s and 5p shells effectively shield inner 4f electrons from significant interaction with the local crystalline field associated with the charges on neighboring ions in solid state configurations.

1. YIG Laser: Nd^{+3} randomly distributed on Y^{+3} sites in Y$_3$Al$_5$O$_{12}$ (garnet)
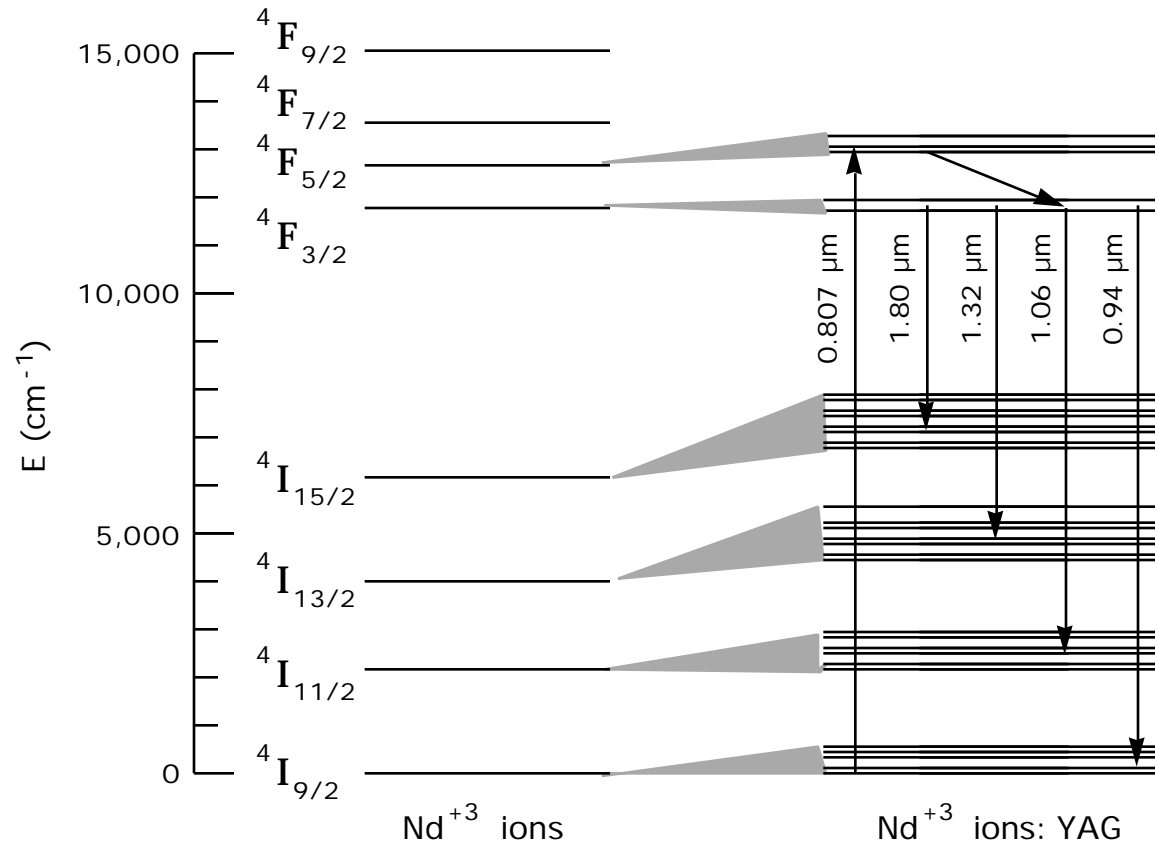
2. EDFA (Erbium Doped Fiber Amplifier): Er^{+3} randomly distributed in glass
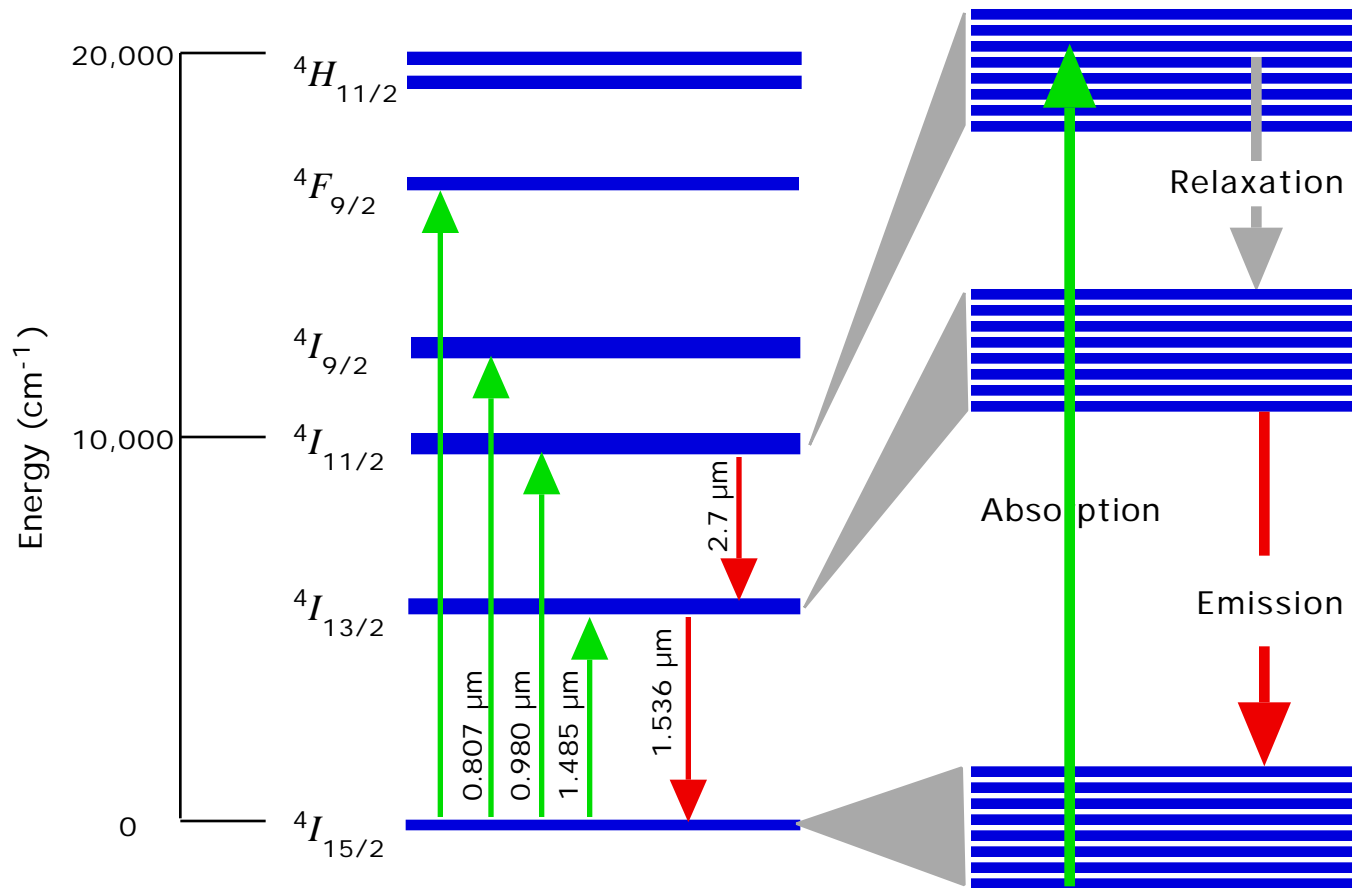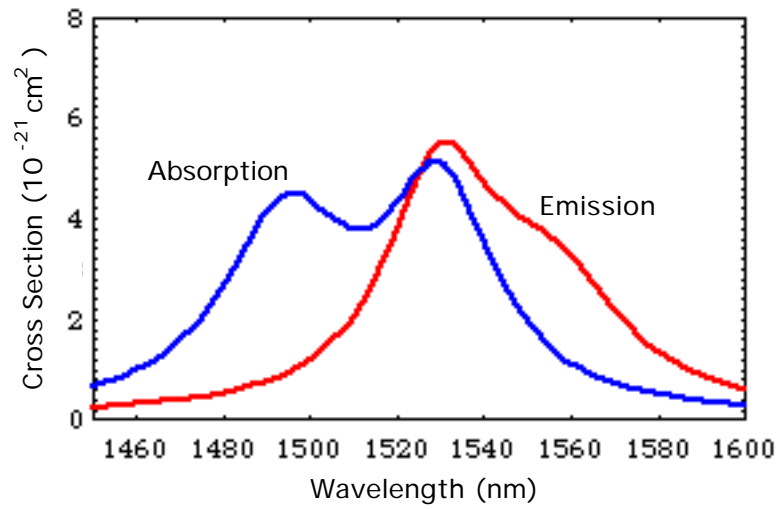
The absortion and emmission cross sections for the $^4I_{15/2}$ → $^4I_{13/2}$ transition in Er^{+3}.