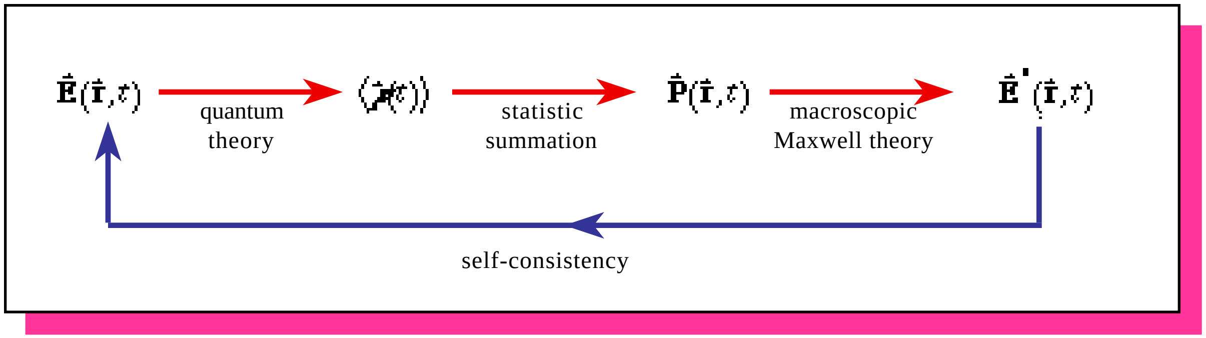
VI. SEMICLASSICAL LASER THEORY [23]

LASER SELF-CONSISTENCY EQUATIONS

Lamb's theory of laser operation provides a very powerful means for interpreting and predicting complex time dependent behavior without invoking all the intricacy of a full quantum mechanical theory. It is semiclassical, self-consistent theory in the following sense:



Suppose that the field in the laser is represented in the form

$$E(z,t) = \frac{1}{2}\sum_{n} \mathcal{E}_n(t)\,\exp\left[-i\left(\omega_n t + \varphi_n(t)\right)\right] U_n(z) + c.c. \qquad \text{[VI-1]}$$

where, for example,

in a simple a cavity laser $\quad U_n(z) \quad \sin k_n z = \sin\left(\frac{n\pi}{L} z\right)$ **standing wave** [VI-2a]

and in a ring laser $\quad U_n(z) \quad \exp\left(i\,k_n\,z\right) = \exp\left(i\frac{n\,2\pi}{L} z\right)$ **traveling wave** [VI-2b]

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23 An adaptation or interpretation of Lamb's "semiclassical" or "self-consistent" laser theory as first presented in *Phys. Rev.* **134**, A1429 (1964) and refined in countless other treatments. In these lecture notes we drawn extensively on M. Sargent III, M. O. Scully and W. E. Lamb, Jr., *Laser Physics*, Addison-Wesley (1974) and P. Meystre and M. Sargent III, *Elements of Quantum Optics*, Springer-Verlag (1992).

With this representation for the field the **induced polarization** can be expressed as

$$P(z,t) = \frac{1}{2}\sum_{n} \mathcal{P}_{n}(t)\,\exp\left[-i\left(\omega_{n}\,t + \varphi_{n}(t)\right)\right] U_{n}(z) + \text{c.c.} \qquad \text{[VI-3]}$$

where $\mathcal{P}_{n}(t)$ is the complex, slowly varying component of the polarization of the *n*th mode. If we take the wave equation in the form[24]

$$-\ {}^{2}\vec{\mathbf{E}} + \mu_{o}\frac{\partial \vec{\mathbf{J}}}{\partial t} + \frac{1}{c^{2}}\frac{\partial^{2}\vec{\mathbf{E}}}{\partial t^{2}} = -\mu_{o}\frac{\partial^{2}\vec{\mathbf{P}}}{\partial t^{2}} \qquad \text{[VI-4]}$$

where second term is included as a means to account for cavity losses. From Equation [VI-1] assuming that $\mathcal{E}_{n}$, $\mathcal{P}_{n}$, and $\dot{\varphi}_{n}$ are slowly varying functions of time[25]

$$\begin{aligned} -\ {}^{2}\vec{\mathbf{E}} \quad - \frac{\partial^{2}E(z,t)}{\partial z^{2}} &= -\frac{1}{2}\sum_{n}\mathcal{E}_{n}\,\exp\left[-i\left(\omega_{n}\,t+\varphi_{n}\right)\right]\frac{\partial^{2}U_{n}}{\partial z^{2}} + c.c. \\ &= \frac{1}{2\,c^{2}}\sum_{n}\Omega_{n}^{2}\,\mathcal{E}_{n}\,\exp\left[-i\left(\omega_{n}\,wt+\varphi_{n}\right)\right]U_{n} + c.c. \end{aligned} \qquad \text{[VI-5a]}$$

where $\Omega_{n} = c\,k_{n}$ are the eigenfrequencies of the ***cold resonator*** eigenmodes.

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \frac{\partial E(z,t)}{\partial t} = \frac{1}{2}\sum_{n}\left\{\dot{\mathcal{E}}_{n} - i\,\mathcal{E}_{n}\left(\omega_{n}+\dot{\varphi}_{n}\right)\right\}\exp\left[-i\left(\omega_{n}\,t+\varphi_{n}\right)\right]U_{n}(z) + c.c. \qquad \text{[VI-5b]}$$

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24 In this formulation we are assuming that $-\ \vec{\mathbf{P}} \approx 0$.

25 The so called ***slowly-varying amplitude and phase approximation*** (SVAP) is used extensively in treating problems in laser dynamics. In the SVAP approximation it is assumed that

$$\left\{\frac{\partial}{\partial z}\ln\left|\mathcal{E}_{n}\right|;\ \frac{\partial}{\partial z}\ln\left|\mathcal{P}_{n}\right|;\ \frac{\partial}{\partial z}\left|\varphi_{n}\right|\right\} << k_{n} \text{ and } \left\{\frac{\partial}{\partial t}\ln\left|\mathcal{E}_{n}\right|;\ \frac{\partial}{\partial t}\ln\left|\mathcal{P}_{n}\right|;\ \frac{\partial}{\partial t}\left|\varphi_{n}\right|\right\} << \Omega_{n}$$

$$\mu_o \frac{\partial \vec{\mathbf{J}}}{\partial t} \quad \mu_o \frac{\partial J(z,t)}{\partial t} \approx \frac{\mu_o \sigma}{2} \sum_n \left\{ -i\, \mathcal{E}_n\, \omega_n \right\} \exp\left[-i\left(\omega_n t + \varphi_n \right) \right] U_n(z) + c.c. \qquad \text{[VI-5c]}$$

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \frac{\partial^2 E\, w(z,t)}{\partial t^2} = \frac{1}{2} \sum_n \left\{ \ddot{\mathcal{E}}_n - i\, 2\, \dot{\mathcal{E}}_n \left(\omega_n + \dot{\varphi}_n \right) + \mathcal{E}_n \left[-\left(\omega_n + \dot{\varphi}_n \right)^2 - i\, \ddot{\varphi}_n \right] \right\} \exp\left[-i\left(\omega_n t + \varphi_n \right) \right] U_n(z) + c.c. \qquad \text{II-5d]}$$

which reduces in the **SVAP** approximation to

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \quad \frac{\partial^2 E(z,t)}{\partial t^2} \approx \frac{1}{2} \sum_n \left\{ -i\, 2\, \dot{\mathcal{E}}_n\, \omega_n + \mathcal{E}_n \left[-\omega_n^2 - 2\omega_n\, \dot{\varphi}_n \right] \right\} \exp\left[-i\left(\omega_n\, wt + \varphi_n \right) \right] U_{nw}(z) + c.c. \qquad \text{[VI-5d']}$$

and from Equation [VI-3]

$$\frac{\partial^2 \vec{\mathbf{P}}}{\partial t^2} \quad \frac{\partial^2 P(z,t)}{\partial t^2} = \frac{1}{2} \sum_n \left\{ \ddot{\mathcal{P}}_n - i\, 2\, \dot{\mathcal{P}}_n \left(\omega_n + \dot{\varphi}_n \right) + \mathcal{P}_n \left[-\left(\omega_n + \dot{\varphi}_n \right)^2 - i\, \ddot{\varphi}_n \right] \right\} \exp\left[-i\left(\omega_n t + \varphi_n \right) \right] U_n(z) + c.c.$$
$$\approx \frac{1}{2} \sum_n \left\{ -\omega_n^2\, \mathcal{P}_n \right\} \exp\left[-i\left(\omega_n t + \varphi_n \right) \right] U_n(z) + c.c. \qquad \text{[VI-6]}$$

If these representations are to be valid when substituted into Equation [VI-4], the following equations must hold:

$$\Omega_n^2\, \mathcal{E}_n - i\left(\frac{\sigma}{\varepsilon_o} \right) \omega_n\, \mathcal{E}_n - i\, 2\omega_n\, \dot{\mathcal{E}}_n - \omega_n^2\, \mathcal{E}_n - 2\omega_n\, \dot{\varphi}_n\, \mathcal{E}_n \approx \frac{1}{\varepsilon_o} \omega_n^2\, \mathcal{P}_n \qquad \text{[VI-7]}$$

We adjust the ***fictional*** conductivity to account for the loss of energy or time decay of the given mode -- *viz.* we take $\sigma = \varepsilon_o\, \omega_n / Q_n$ where Q_n is the "Q" of the mode. Equating real and imaginary parts of Equation [VI-7] we obtain Lamb's **master laser self-consistency equations** -- *viz.*

$$\dot{\mathcal{E}}_n + \left(\frac{\omega_n}{2Q_n}\right)\mathcal{E}_n \approx -\frac{\omega_n}{2\varepsilon_o}\,\mathcal{Im}\left\{\mathcal{P}_n\right\} \qquad \text{[VI-8a]}$$

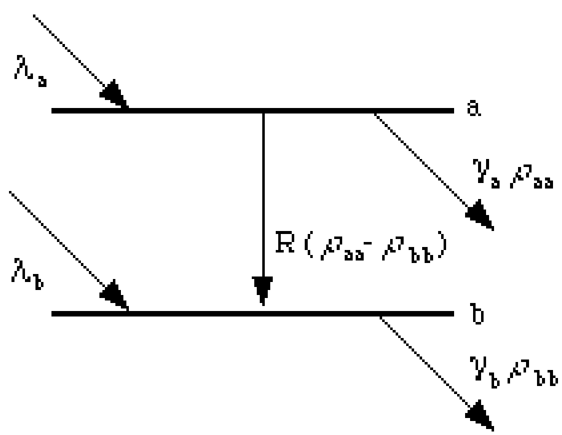
$$\Omega_n^2\,\mathcal{E}_n - \omega_n^2\,\mathcal{E}_n - 2\omega_n\,\dot{\varphi}_n\,\mathcal{E}_n \approx \frac{1}{\varepsilon_o}\omega_n^2\,\mathcal{Re}\left\{\mathcal{P}_n\right\} \qquad \text{[VI-8b]}$$

or

$$\omega_n + \dot{\varphi}_n \approx \Omega_n - \frac{\omega_n}{2\varepsilon_o}\,\mathcal{Re}\left\{\mathcal{P}_n\right\}\Big/\mathcal{E}_n \qquad \text{[VI-8b']}$$

POLARIZATION OF THE MEDIUM:

The simplest application of Lamb's theory assumes that the laser medium consists of an ensemble of two-level atoms with a single well-defined transition $\hbar\omega_{ab}$ which is **homogeneously broaden.** Possible "pump" and "decay" processes are indicated schematically in the following figure:



The model describes, in effect, a so called ***four-level*** **homogeneously broaden laser** where both levels relaxed to some presumed *ground state* and are "pumped" via higher energy excited states. $\rho\left(s,z,t',t\right)$ is defined as the density matrix operator at a place z and a time t which is associated with atoms excited to the state s (where $s = a$ or b) at a time t' and $\lambda_s\left(z,t'\right)$ as the rate at which atoms are excited to the state s (per atom, per second per unit volume). The macroscopic polarization is, thus, given by

$$P\left(z,t\right) = \sum_{sw} \int_{-\infty}^{t} dt' \, \lambda_s\left(z,t'\right) \langle e \, \mathcal{D}isp \rangle$$
$$= \sum_{s} \int_{-\infty}^{t} dt' \, \lambda_s\left(z,t'\right) \rho_{ab}\left(s,z,t',t\right) + c.c. \qquad \text{[VI-9]}$$

If we project this expression on to the modes of the laser cavity and compare with Equation [VI-3], we see that

$$\mathcal{P}_n(t) = 2 \quad \exp\left[\, i\left(\omega_n t + \varphi_n(t)\right)\right] \frac{1}{\mathcal{M}_n} \int_0^L dz U_n(z) \sum_s \int_{-\infty}^{t} dt' \, \lambda_s\left(z,t'\right) \rho_{ab}\left(s,z,t',t\right) \qquad \text{[VI-10a]}$$

where $\mathcal{M}_n = \int_0^L dz \left| U_n(z) \right|^2$ is the mode normalization factor. To facilitated the integration of this equation, we re-write it in terms of the **population matrix operator** $\bar{\rho}\left(z,t\right)$ -- *viz.*

$$\mathcal{P}_n(t) = 2 \quad \exp\left[\, i\left(\omega_n t + \varphi_n(t)\right)\right] \frac{1}{\mathcal{M}_n} \int_0^L dz \, U_n(z) \, \bar{\rho}_{ab}\left(z,t\right) \qquad \text{[VI-10b]}$$

where

$$\bar{\rho}\left(z,t\right) \equiv \sum_s \int_{-\infty}^{t} dt' \, \lambda_s\left(z,t'\right) \rho\left(s,z,t',t\right) \, . \qquad \text{[VI-11]}$$

The equation of motion of this population matrix is found by differentiating its defining equation -- *i.e.*

$$\frac{d\bar{\rho}(z,t)}{d\,t} = \sum_s \lambda_s(z,t)\,\rho(s,z,t,t) + \sum_s \int_{-\infty}^{t} d\,t'\,\lambda_s(z,t')\,\dot{\rho}(s,z,t',t)\;. \qquad \text{[VI-12]}$$

By definition $\rho_{\alpha\beta}(s,z,t,t) = \delta_{\alpha s}\delta_{\beta s}$, so that first term on the right-hand-side is replaced by the matrix

$$\begin{bmatrix} \lambda_a(z,t) & 0 \\ 0 & \lambda_b(z,t) \end{bmatrix}$$

If we interpret the $\dot{\rho}(s,z,t',t)$ components of the second term on the right-hand-side of Equation [VI-12], as the time derivatives of the ***pure state*** density matrix components given by Equations [III-14a] and [III-14b] of this set of lectures notes,[26] the component equations of motion for the population matrix $\bar{\rho}(z,t)$ become

$$\dot{\bar{\rho}}_{aa} = \lambda_a - \gamma_a\,\bar{\rho}_{aa} - \left[i\,\hbar^{-1}\mathcal{V}_{ab}\,\bar{\rho}_{ab} + c.c.\right] \qquad \text{[VI-13a]}$$

$$\dot{\bar{\rho}}_{bb} = \lambda_b - \gamma_b\,\bar{\rho}_{bb} + \left[i\,\hbar^{-1}\mathcal{V}_{ab}\,\bar{\rho}_{ab} + c.c.\right] \qquad \text{[VI-13b]}$$

$$\dot{\bar{\rho}}_{ab} = -\left(i\,\omega_{ab} + \gamma_{ab}\right)\bar{\rho}_{ab} + i\,\hbar^{-1}\mathcal{V}_{ab}\left[\bar{\rho}_{aa} - \bar{\rho}_{bb}\right] \qquad \text{[VI-13c]}$$

Formal integration of Equation [VI-13c] yields

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[26] Including damping, Equations [III-16a] and [III-16b] of this set of lecture notes would have the form

$$\dot{\rho}_{aa} = -\gamma_a\,\rho_{aa} - \left[i\,\hbar^{-1}\mathcal{V}_{ab}\,\rho_{ab} + c.c.\right]$$
$$\dot{\rho}_{bb} = -\gamma_b\,\rho_{bb} + \left[i\,\hbar^{-1}\mathcal{V}_{ab}\,\rho_{ab} + c.c.\right]$$
$$\dot{\rho}_{ab} = -\left(i\,\omega_{ab} + \gamma_{ab}\right)\rho_{ab} + i\,\hbar^{-1}\mathcal{V}_{ab}\left[\rho_{aa} - \rho_{bb}\right]$$

$$\rho_{ab} = i\hbar^{-1}\int_{-\infty}^{t} dt' \exp w\left[-\left(i\,\omega_{ab}+\gamma_{ab}\right)\left(t-t'\right)\right]\mathcal{V}_{ab}\left(z,t'\right)\left[\rho_{aa}\left(z,t'\right)-\rho_{bb}\left(z,t'\right)\right] \quad \text{[VI-14]}$$

SINGLE MODE OPERATION:

For a **single-mode standing wave field** in the **rotating-wave approximation**

$$\mathcal{V}_{ab}\left(z,t'\right) = -\frac{1}{2}\quad \mathcal{E}_{n}(t)\exp\left[-i\left(\omega_{n}t'+\varphi_{n}\left(t'\right)\right)\right]U_{n}(z) \qquad \text{[VI-15]}$$

so that

$$\begin{aligned}\rho_{ab}(z,t) &= -\frac{1}{2}i\quad \hbar^{-1}\exp\left[-i\,\omega_{n}t\right]U_{n}(z)\int_{-\infty}^{t} dt' \exp\left\{-\left[i\left(\omega_{ab}-\omega_{n}\right)+\gamma_{ab}\right]\left(t-t'\right)-i\,\varphi_{n}\left(t'\right)\right\}\mathcal{E}_{n}\left(t'\right)\left[\rho_{aa}\left(z,t'\right)-\rho_{bb}\left(z,t'\right)\right]\\ &= -\frac{1}{2}i\quad \hbar^{-1}\exp\left[-i\,\omega_{n}t\right]U_{n}(z)\int_{0}^{+\infty} d\tau \exp\left\{-\left[i\left(\omega_{ab}-\omega_{n}\right)+\gamma_{ab}\right]\tau-i\,\varphi_{n}(t-\tau)\right\}\mathcal{E}_{n}(t-\tau)\left[\rho_{aa}(z,t-\tau)-\rho_{bb}(z,t-\tau)\right]\end{aligned} \quad \text{[VI-16]}$$

The integration can be done simply, if the changes in $\mathcal{E}_{n}$, φ_{n}, and $\rho_{aa}-\rho_{bb}$ are negligible in a time γ_{ab}^{-1} so that

$$\begin{aligned}\rho_{ab}(z,t) &= -\frac{1}{2}i\quad \hbar^{-1}\mathcal{E}_{n}(t)\exp\left[-i\left(\omega_{n}t+\varphi_{n}(t)\right)\right]U_{n}(z)\left[\rho_{aa}(z,t)-\rho_{bb}(z,t)\right]\left[i\left(\omega_{ab}-\omega_{n}\right)+\gamma_{ab}\right]^{-1}\\ &= -\frac{1}{2}i\quad \hbar^{-1}\mathcal{E}_{n}(t)\exp\left[-i\left(\omega_{n}t+\varphi_{n}(t)\right)\right]U_{n}(z)\left[\rho_{aa}(z,t)-\rho_{bb}(z,t)\right]\mathcal{D}\left(\omega_{ab}-\omega_{n};\gamma_{ab}\right)\end{aligned} \quad \text{[VI-17]}$$

where $\mathcal{D}(u-v;w)\equiv\left[i\,(u-v)+w\right]^{-1}$ represents the so called **complex Lorentzian denominator**. Substituting this result into Equations [VI-13a] and [VI-13b] we find the previously discussed **rate equation approximation** for the population components -- *viz*.

$$\begin{aligned}\dot{\bar\rho}_{aa} &= \lambda_a - \gamma_a\,\bar\rho_{aa} - R\left(\bar\rho_{aa} - \bar\rho_{bb}\right)\\ \dot{\bar\rho}_{bb} &= \lambda_b - \gamma_b\,\bar\rho_{bb} + R\left(\bar\rho_{aa} - \bar\rho_{bb}\right)\end{aligned} \qquad \text{[VI-18]}$$

with the **rate constant** is given by

$$\begin{aligned}R &= \frac{1}{2}\left(\frac{\wp}{\hbar}\right)^2 \mathcal{E}_n^2(t)\,|\,U_n(z)\,|^2\,\gamma_{ab}\left[\left(\omega_{ab} - \omega_n\right)^2 + \gamma_{ab}^2\right]^{-1}\\ &= \frac{1}{2}\left(\frac{\wp}{\hbar}\right)^2 \mathcal{E}_n^2(t)\,|\,U_n(z)\,|^2\,\gamma_{ab}^{-1}\,\mathcal{L}\left(\omega_{ab} - \omega_n;\gamma_{ab}\right)\end{aligned} \qquad \text{[VI-19]}$$

where $\mathcal{L}(u - v;\, w) \equiv w^2\left[(u - v)^2 + w^2\right]^{-1}$ represents the **dimensionless Lorentzian function**.

For steady state -- *i.e.* $\dot{\bar\rho}_{aa} = \dot{\bar\rho}_{bb} = 0$

$$\bar\rho_{aa} - \bar\rho_{bb} = \frac{\lambda_a\,\gamma_a^{-1} - \lambda_b\,\gamma_b^{-1}}{1 + R(z)\Big/\left(\dfrac{\gamma_a\,\gamma_b}{\gamma_a + \gamma_b}\right)} = \frac{N(z)}{1 + R(z)/R_S} \qquad \text{[VI-20]}$$

where $N(z)$ is the unsaturated population difference and $R_S = \gamma_a\,\gamma_b\left(\gamma_a + \gamma_b\right)^{-1}$. Combining Equations [VI-10b], [VI-17] and [VI-20] we obtain

$$\mathcal{P}_n(t) = -\ {}^2\,\hbar^{-1}\mathcal{E}_n\ \mathcal{D}\left(\omega_{ab} - \omega_n;\gamma_{ab}\right)\frac{1}{\mathcal{M}_n}\int_0^L dz\,\frac{N(z)\,|U_n(z)|^2}{1 + R(z)/R_S} \qquad \text{[VI-21]}$$

Although this result **can be integrated exactly**,[27] there is more insight to be gained by expanding the ***saturation denominator*** in powers of $R(z)/R_S$ -- *viz.*

$$\mathcal{P}_n(t) = -\ {}^2\,\hbar^{-1}\mathcal{E}_n\ \mathcal{D}(\omega_{ab}-\omega_n;\gamma_{ab})\,\frac{1}{\mathcal{M}_n}\int_0^L dz\ N(z)\,|U_n(z)|^2 \sum_{k=0}^{\infty}\left(\frac{R(z)}{R_S}\right)^k \qquad \text{[VI-22a]}$$

To second order in $R(z)/R_S$ we obtain

$$\mathcal{P}_n(t) \approx -\ {}^2\,\hbar^{-1}\mathcal{E}_n\ \mathcal{D}(\omega_{ab}-\omega_n;\gamma_{ab})\ \overline{N}\left\{1-\frac{3}{4}\,I_n\left(\frac{\gamma_a+\gamma_b}{\gamma_{ab}}\right)\mathcal{L}(\omega_{ab}-\omega_n;\gamma_{ab})\right\} \qquad \text{[VI-22b]}$$

where

$$\overline{N}(t) \equiv \frac{1}{\mathcal{M}_n}\int_0^L dz\ N(z,t)\,|U_n(z)|^2 \qquad \text{[VI-23a]}$$

and

$$I_n = \frac{1}{2}\,\frac{{}^2\,\mathcal{E}_n^{\,2}}{\hbar^2}\,\frac{1}{\gamma_a\gamma_b}\,. \qquad \text{[VI-23b]}$$

is the dimensionless mode intensity.

To complete the self-consistency loop, we combine our expression for the complex mode polarization, Equation [VI-22b], and the self-consistency conditions, Equations [VI-8].

From Equation [VI-8a]

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27 Exactly $$\mathcal{P}_n(t) = -\ {}^2\,\hbar^{-1}\mathcal{E}_n\ \mathcal{D}(\omega_{ab}-\omega_n;\gamma_{ab})\ \overline{N}\ f(w)$$

where $$f(w) = \frac{2}{w}\left[1-(1+w)^{-1/2}\right] \quad \text{and} \quad w = I_n\left(\frac{\gamma_a+\gamma_b}{\gamma_{ab}}\right)\mathcal{L}(\omega_{ab}-\omega_n;\gamma_{ab})$$

$$\dot{\mathcal{E}}_n + \left(\frac{\omega_n}{2Q_n}\right)\mathcal{E}_n = -\frac{\omega_n}{2\varepsilon_o}\mathcal{Im}\left\{\mathcal{P}_n\right\}$$

$$\frac{1}{2}\omega_n \frac{\wp^2\,\mathcal{E}_n}{\hbar\varepsilon_o}\left(\frac{1}{\gamma_{ab}}\right)\mathcal{L}\left(\omega_{ab}-\omega_n;\gamma_{ab}\right)\bar{N}\left\{1-\frac{3}{4}I_n\left(\frac{\gamma_a+\gamma_b}{\gamma_{ab}}\right)\mathcal{L}\left(\omega_{ab}-\omega_n;\gamma_{abw}\right)\right\}. \qquad \text{[VI-24]}$$

Multiplying this equation by $\dfrac{\wp^2\,\mathcal{E}_n}{2\hbar\,\gamma_b\gamma_b}$, we obtain

$$\dot{I}_n \approx 2I_n\left\{-\left(\frac{\omega_n}{2Q_n}\right)+\frac{1}{2}\omega_n\frac{\wp^2}{\hbar\varepsilon_o}\left(\frac{1}{\gamma_{ab}}\right)\mathcal{L}\left(\omega_{ab}-\omega_n;\gamma_{ab}\right)\bar{N}\left[1-\frac{3}{4}I_n\left(\frac{\gamma_a+\gamma_b}{\gamma_{ab}}\right)\mathcal{L}\left(\omega_{ab}-\omega_{nw};\gamma_{ab}\right)\right]\right\} \qquad \text{[VI-25a]}$$

This nonlinear equation of motion may be written to advantage as

$$\dot{I}_n \approx 2I_n\left[\alpha_n - \beta_n I_n\right] \qquad \text{[VI-25b]}$$

where the ***linear gain*** is measured by

$$\alpha_n = \frac{1}{2}\omega_n\frac{\wp^2}{\hbar\varepsilon_o}\left(\frac{1}{\gamma_{ab}}\right)\mathcal{L}\left(\omega_{ab}-\omega_n;\gamma_{ab}\right)\bar{N} - \left(\frac{\omega_n}{2Q_n}\right) \qquad \text{[VI-26a]}$$

and ***nonlinear saturation*** by

$$\beta_n = \frac{3}{8}\bar{N}\frac{\omega_n\,\wp^2}{\hbar\varepsilon_o\,\gamma_{ab}}\left(\frac{\gamma_a+\gamma_b}{\gamma_{ab}}\right)\mathcal{L}^2\left(\omega_{ab}-\omega_n;\gamma_{ab}\right) \qquad \text{[VI-26b]}$$

From Equation [VI-8b'], we see that

$$\omega_n + \dot{\varphi}_n \approx \Omega_n - \frac{\omega_n}{2\varepsilon_o}\mathcal{Re}\left\{\mathcal{P}_n\right\}/\mathcal{E}_n$$

Thus

$$\omega_n + \dot{\varphi}_n \quad \Omega_n + \frac{1}{2}\,\omega_n \frac{{}^{2}}{\hbar\varepsilon_o} \frac{\left(\omega_{ab} - \omega_n\right)}{\gamma_{ab}^2} \mathcal{L}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right) \bar{N} \left\{ 1 - \frac{3}{4} I_n \left(\frac{\gamma_a + \gamma_b}{\gamma_{ab}}\right) \mathcal{L}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right) \right\} \qquad \text{[VI-27]}$$

$$\Omega_n + \sigma_n - \beta_n I_n$$

where the linear mode ***pulling effect*** is measured by

$$\sigma_n = \frac{1}{2}\,\bar{N}\,\frac{\omega_n \;{}^{2}}{\hbar\varepsilon_o\,\gamma_{ab}} \frac{\left(\omega_{ab} - \omega_n\right)}{\gamma_{ab}} \mathcal{L}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right) \qquad \text{[VI-28a]}$$

and nonlinear ***pushing effect*** by

$$\beta_n = \frac{3}{4}\,\bar{N}\,\frac{\omega_n \;{}^{2}}{\hbar\varepsilon_o\,\gamma_{ab}} \frac{\left(\omega_{ab} - \omega_n\right)}{\gamma_{ab}} \left(\frac{\gamma_a + \gamma_b}{\gamma_{ab}}\right) \left[\mathcal{L}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right)\right]^2 \qquad \text{[VI-28b]}$$

From Equation [VI-25a], we find the **"at-resonance"threshold condition to be:**

$$\left(\frac{\omega_n}{2Q_n}\right) = \frac{1}{2}\,\omega_n \frac{{}^{2}}{\hbar\varepsilon_o} \left(\frac{1}{\gamma_{ab}}\right) \mathcal{L}\left(0; \gamma_{ab}\right) \bar{N}_{Th} \qquad \frac{1}{Q_n} = \frac{{}^{2}\;\bar{N}_{Th}}{\hbar\varepsilon_o\,\gamma_{ab}} \qquad \text{[VI-29]}$$

where $\bar{N}_{Th}$ is the required population inversion at threshold.

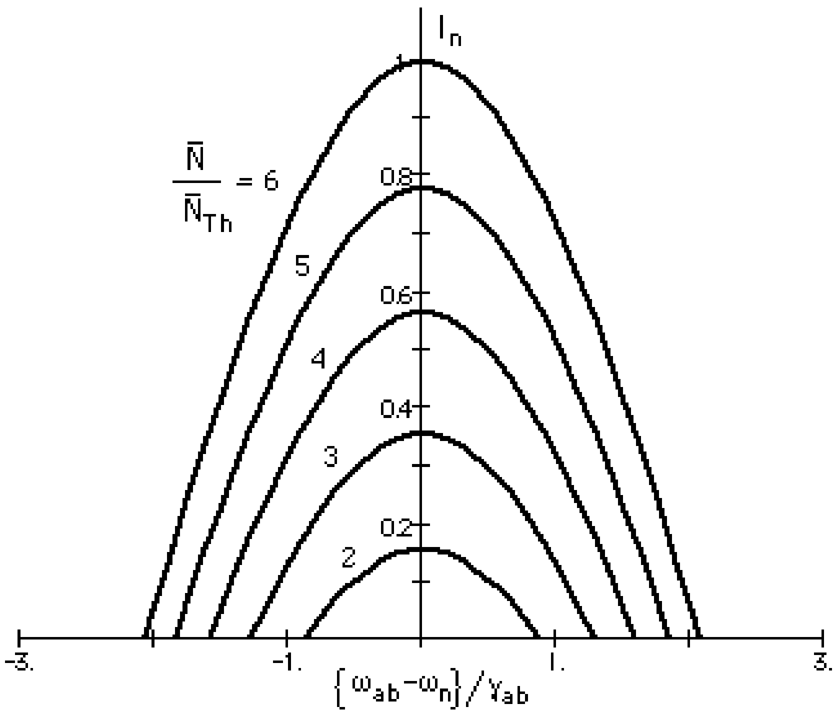
For the **general steady state condition**, we have

$$I_n = \frac{1}{2}\,\frac{{}^{2}\;\mathcal{E}_n^{\;2}}{\hbar^2}\,\frac{1}{\gamma_a\,\gamma_b} \approx \alpha_n / \beta_n \qquad \text{[VI-30a]}$$

or

$$I_n = \frac{4}{3}\left(\frac{\gamma_{ab}}{\gamma_a + \gamma_b}\right)\frac{\mathcal{L}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right) - \bar{N}_{Th}/\bar{N}}{\mathcal{L}^2\left(\omega_{ab} - \omega_n; \gamma_{ab}\right)} \qquad \text{[VI-30b]}$$

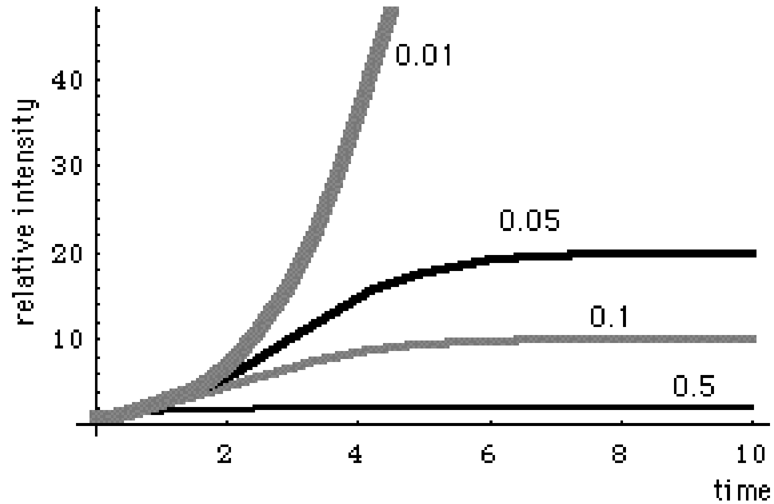
SINGLE-MODE, STANDING-WAVE LASER INTENSITY



The **time dependent intensity build-up** follows from integrating Equation [VI-25b] to obtain

$$I_n(t) = \frac{\alpha_n\, I_n(0)\exp\left(\alpha_n\, t\right)}{\alpha_n + \beta_n\, I_n(0)\left[\exp\left(\alpha_n\, t\right) - 1\right]} \qquad \text{[VI-29]}$$

TRANSIENT INTENSITY BUILDUP



MULTIMODE OPERATION:

When we consider the possibility that two or more laser modes may be simultaneously excited or oscillating, things get quite a bit more complicated and we must refine our analysis. In particular, we must treat the time dependence of the population difference $\bar{\rho}_{aa} - \bar{\rho}_{bb}$ with greater care since variations in the population difference tend to modulate and couple the possible modal excitations.

Reflection on the form of Equations [VI-13a] and [VI-13b] in the absence of modal excitation, suggests that we take

$$N(z,t) \equiv \left[\bar{\rho}_{aa}^{(0)}(z,t) - \bar{\rho}_{bb}^{(0)}(z,t)\right] = \lambda_a \gamma_a^{-1} - \lambda_b \gamma_b^{-1} \qquad \text{[VI-31]}$$

as the zeroth-order approximation to the time dependence of the population difference. With finite excitation of one or more modes we may write in the rotating-wave approximation

$$\mathcal{V}_{ab}(z,t) = -\frac{1}{2} \sum_{n} \mathcal{E}_n(t) \exp\left[-i\left(\omega_n t + \varphi_n(t)\right)\right] U_n(z) \qquad \text{[VI-32]}$$

and use Equation [VI-14] to obtain the following first-order approximation to the time dependence of the off-diagonal component of the population matrix:

$$\begin{aligned}
\rho_{ab}^{(1)} &= i\,\hbar^{-1} \int_{-\infty}^{t} dt' \exp\left[-\left(i\,\omega_{ab} + \gamma_{ab}\right)(t-t')\right] \mathcal{V}_{ab}(z,t')\left[\rho_{ab}^{(0)}(z,t) - \rho_{ab}^{(0)}(z,t)\right] \\
&= -\frac{1}{2} i \quad \hbar^{-1} N(z,t) \sum_{n} \mathcal{E}_n(t) \exp\left[-i\left(\omega_n t + \varphi_n(t)\right)\right] U_n(z) \int_{-\infty}^{t} dt' \exp\left\{-\left[i\left(\omega_{ab} - \omega_n\right) + \gamma_{ab}\right](t-t')\right\} \qquad \text{[VI-33]} \\
&= -\frac{1}{2} i \quad \hbar^{-1} N(z,t) \sum_{n} \mathcal{E}_n(t) \exp\left[-i\left(\omega_n t + \varphi_n(t)\right)\right] U_n(z)\, \mathcal{D}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right)
\end{aligned}$$

In carrying out this integration we have, again, assumed that changes in $\mathcal{E}_n$, and φ_{φ_n}, are negligible in a time γ_{ab}^{-1}. Using this result and Equation [VI-13a] we obtain the time derivative of the second-order approximation to the time dependence of the population of the upper level -- *viz*.

$$\begin{aligned}
\dot{\rho}_{aa} \quad & \dot{\rho}_{aa}^{(0)} + \dot{\rho}_{aa}^{(2)} \\
&= \lambda_a - \gamma_a \rho_{aa} - \left[\left|\frac{\wp}{2\hbar}\right|^2 N(z,t) \sum_{n}\sum_{m} \mathcal{E}_n \mathcal{E}_m \exp\left\{i\left[\left(\omega_n - \omega_m\right)t + \varphi_n - \varphi_m\right]\right\} U_m(z)\, U_n(z)\, \mathcal{D}\left(\omega_{ab} - \omega_m; \gamma_{ab}\right) + c.c.\right] \qquad \text{[VI-34]}
\end{aligned}$$

The key point here is that the population ρ_{aa} has ***pulsations* at the intermode beat frequencies!** Integrating we find

$$\rho_{aa}^{(2)} = -\left|\frac{}{2\hbar}\right|^2 N(z,t) \sum_n \sum_m \mathcal{E}_n \mathcal{E}_m \exp\left\{ i\left[\left(\omega_n - \omega_m\right) t + \varphi_n - \varphi_m\right]\right\} U_m(z)\, U_n(z)\, \mathcal{D}\left(\omega_n - \omega_m; \gamma_a\right) \mathcal{D}\left(\omega_{ab} - \omega_m; \gamma_{ab}\right) + c.c. \quad \text{[VI-35]}$$

and since $\rho_{bb}^{(2)} = -\rho_{aa}^{(2)}$ with $\gamma_a \quad \gamma_b$ we obtain the second-order approximation to the population difference as

$$\begin{aligned} \rho_{aa}^{(2)} - \rho_{bb}^{(2)} = -\left|\frac{}{2\hbar}\right|^2 N(z,t) \sum_n \sum_m \mathcal{E}_n \mathcal{E}_m \exp\left\{ i\left[\left(\omega_n - \omega_m\right) t + \varphi_n - \varphi_m\right]\right\} U_m(z)\, U_n(z) \\ \times\left[\mathcal{D}\left(\omega_n - \omega_m; \gamma_a\right) + \mathcal{D}\left(\omega_n - \omega_m; \gamma_b\right)\right]\left[\mathcal{D}\left(\omega_{ab} - \omega_m; \gamma_{ab}\right) + \mathcal{D}\left(\omega_n - \omega_{ab}; \gamma_{ab}\right)\right] \end{aligned} \quad \text{[VI-36]}$$

Substituting this expression into the formal integral of Equation [VI-14], we find, directly, the third-order approximation to the off-diagonal component of the population matrix

$$\begin{aligned} \rho_{ab}^{(3)} = \frac{1}{8}\, i \left|\frac{}{2\hbar}\right|^2 N(z,t) \sum_l \sum_m \sum_n \mathcal{E}_l \mathcal{E}_m \mathcal{E}_n\, U_l(z)\, U_m(z)\, U_n(z) \\ \times \exp\left\{ -i\left[\left(\omega_l - \omega_m + \omega_n\right) t + \varphi_l - \varphi_m + \varphi_n\right]\right\} \mathcal{D}\left(\omega_{ab} - \omega_l + \omega_m - \omega_n; \gamma_{ab}\right) \\ \times\left[\mathcal{D}\left(\omega_m - \omega_n; \gamma_a\right) + \mathcal{D}\left(\omega_m - \omega_n; \gamma_b\right)\right]\left[\mathcal{D}\left(\omega_{ab} - \omega_n; \gamma_{ab}\right) + \mathcal{D}\left(\omega_m - \omega_{ab}; \gamma_{ab}\right)\right] \end{aligned} \quad \text{[VI-37]}$$

Using Equation [VI-10b] and invoking Equations [VI-8] the general self-consistency conditions may be written [28]

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[28] It is important to note, that use of Equation [VI-10b] requires evaluation of the following integral :

$$N_{klmn} = \int_0^L dz\, N(z)\, U_k(z)\, U_l(z)\, U_m(z)\, U_n(z) \Big/ \int_0^L dz \left|U_k(z)\right|^2$$

$$\dot{\mathcal{E}}_{\mathrm{k}} = a_{\mathrm{k}}\,\mathcal{E}_{\mathrm{k}} - \sum_{\mathrm{l}}\sum_{\mathrm{m}}\sum_{\mathrm{n}} \mathcal{E}_{\mathrm{l}}\mathcal{E}_{\mathrm{m}}\mathcal{E}_{\mathrm{n}}\ \mathrm{Im}\left\{\vartheta_{\mathrm{klmn}}\ \exp\left[\,i\,\Psi_{\mathrm{klmn}}\right]\right\} \qquad [\text{ VI-38 }]$$

$$\omega_{\mathrm{k}} + \dot{\varphi}_{\mathrm{k}} = \Omega_{\mathrm{k}} + \sigma_{\mathrm{k}} - \sum_{\mathrm{l}}\sum_{\mathrm{m}}\sum_{\mathrm{n}} \mathcal{E}_{\mathrm{k}}^{-1}\mathcal{E}_{\mathrm{l}}\mathcal{E}_{\mathrm{m}}\mathcal{E}_{\mathrm{n}}\ \mathrm{Re}\left\{\vartheta_{\mathrm{klmn}}\ \exp\left[\,i\,\Psi_{\mathrm{klmn}}\right]\right\} \qquad [\text{ VI-39 }]$$

with the coefficients summarized in the following table for a **standing wave** configuration:

Coefficients	Significance
$a_{\mathrm{k}} = \mathcal{L}\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{k}};\, \gamma_{\mathrm{ab}}\right) F_1 - \frac{1}{2}\,\omega_{\mathrm{k}}/Q_{\mathrm{k}}$	**Linear net gain**
$\sigma_{\mathrm{k}} = \left[\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{k}}\right)/\gamma_{\mathrm{ab}}\right] \mathcal{L}\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{k}};\, \gamma_{\mathrm{ab}}\right) F_1$	**Linear mode pulling**
$\beta_{\mathrm{k}} = \mathcal{L}^2\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{k}};\, \gamma_{\mathrm{ab}}\right) F_3$	**Self saturation**
$\vartheta_{\mathrm{klmn}} = i\,\frac{\gamma_{ab}}{4}\left| \quad /2\hbar\right|^2 F_1\left\{1 + \left[N_{2(\mathrm{m-n})} + N_{2(\mathrm{m-l})}\right]/\bar{N}\right\} \mathcal{D}\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{l}} + \omega_{\mathrm{m}} - \omega_{\mathrm{n}};\, \gamma_{\mathrm{ab}}\right)$ $\times\left[\mathcal{D}\left(\omega_{\mathrm{ab}} - \omega_{\mathrm{n}};\, \gamma_{\mathrm{ab}}\right) + \mathcal{D}\left(\omega_{\mathrm{m}} - \omega_{\mathrm{ab}};\, \gamma_{\mathrm{ab}}\right)\right]\left[\mathcal{D}\left(\omega_{\mathrm{m}} - \omega_{\mathrm{n}};\, \gamma_{\mathrm{a}}\right) + \mathcal{D}\left(\omega_{\mathrm{m}} - \omega_{\mathrm{n}};\, \gamma_{\mathrm{b}}\right)\right]$	**General saturation term**

Thus, the coefficients in Equations [VI-38] and [VI-39] are sensitive to the particulars of the laser mode configuration. Since the modal excitations cannot change rapidly , the only integrals of importance are those for which $\omega_k - \omega_l + \omega_m - \omega_n \approx 0$

$$\vartheta_{kkmm}+\vartheta_{kmmk} = i\,\frac{1}{2}\left[\frac{\gamma_a+\gamma_b}{\gamma_a\gamma_b}\right] \quad /2\hbar|^2\, F_1\left\{2+N_{2(m-k)}/\bar{N}\right\}\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)$$
$$\times\left\{\mathcal{L}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)\mathcal{L}\left(\omega_{ab}-\omega_m;\gamma_{ab}\right)\right.$$
$$+\frac{\gamma_{ab}}{2}\left[\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\right]\left[\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)+\mathcal{D}\left(\omega_m-\omega_{ab};\gamma_{ab}\right)\right]$$
$$\left.\times\left[\mathcal{D}\left(\omega_m-\omega_k;\gamma_a\right)+\mathcal{D}\left(\omega_m-\omega_k;\gamma_b\right)\right]\right\}$$

Stationary coefficient

$$\theta_{km} = \left[\left(2\hbar^2\gamma_a\gamma_b\right)/|\quad|^2\right]\,\mathrm{Im}\left(\vartheta_{kkmm}+\vartheta_{kmmk}\right)$$
$$= \frac{1}{3}F_3\left\{2+N_{2(m-k)}/\bar{N}\right\}\left[\mathcal{L}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)\mathcal{L}\left(\omega_{ab}-\omega_m;\gamma_{ab}\right)\right.$$
$$+\frac{1}{2}\left[\gamma_{ab}\right]^2\left[\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\right]\mathrm{Re}\left\{\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)\left[\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)+\mathcal{D}\left(\omega_m-\omega_{ab};\gamma_{ab}\right)\right]\right.$$
$$\left.\left.\times\left[\mathcal{D}\left(\omega_m-\omega_k;\gamma_a\right)+\mathcal{D}\left(\omega_m-\omega_k;\gamma_b\right)\right]\right\}\right]$$

Cross-saturation

$$\Psi_{klmn} = \left(\omega_k-\omega_l+\omega_m-\omega_n\right)t+\varphi_k-\varphi_l+\varphi_m-\varphi_n$$

Relative phase angle

$$\rho_m = \left(\omega_{ab}-\omega_m\right)/\gamma_{ab}\,\mathcal{L}^2\left(\omega_{ab}-\omega_m;\gamma_{ab}\right)F_3$$

Self mode pushing

$$\tau_{km} = \left[\left(2\hbar^2\gamma_a\gamma_b\right)/|\quad|^2\right]\,\mathrm{Re}\left(\vartheta_{kkmm}+\vartheta_{kmmk}\right)$$
$$= \frac{1}{3}F_3\left\{2+N_{2(m-k)}/\bar{N}\right\}\left[\left(\omega_{ab}-\omega_k\right)/\gamma_{ab}\;\mathcal{L}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)\mathcal{L}\left(\omega_{ab}-\omega_m;\gamma_{ab}\right)\right.$$
$$-\frac{1}{2}\left[\gamma_{ab}\right]^2\left[\frac{\gamma_a\gamma_b}{\gamma_a+\gamma_b}\right]\mathrm{Im}\left\{\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)\left[\mathcal{D}\left(\omega_{ab}-\omega_k;\gamma_{ab}\right)+\mathcal{D}\left(\omega_m-\omega_{ab};\gamma_{ab}\right)\right]\right.$$

Cross pushing

$F_1 = \omega_n \quad {}^2\, \overline{N}\, \left[2\,\hbar\,\varepsilon_o\, \gamma_{ab}\right]^{-1}$ $\qquad F_3 = \left(3/2\right) \dfrac{1}{\gamma_{ab}} \left[\dfrac{\gamma_a\,\gamma_b}{\gamma_a + \gamma_b}\right] F_1$	**First-order factor** and **Third-order factor**
$N_{2m} = \dfrac{1}{L}\displaystyle\int dz\; N\left(z\right)\,\cos\left[\left(\dfrac{2\pi m}{L}\right) z\right]$	**Spatial factor**
$\mathcal{D}\left(u - v; w\right) \equiv \left[\,i\,\left(u - v\right) + w\right]^{-1}$	**Lorentzian denominator**
$\mathcal{L}\left(u - v; w\right) \equiv w^2\,\left[\left(u - v\right)^2 + w^2\right]^{-1}$	**Dimensionless Lorentzian**

Two-Mode Operation: For two modes the amplitude determining equations -- *i.e.* Equation [VI-38] - reduce to

$$\dot{\mathcal{E}}_1 = \mathcal{E}_1\left[a_1 - \beta_1\, I_1 - \theta_{12}\; I_2\right]$$
$$\dot{\mathcal{E}}_2 = \mathcal{E}_2\left[a_2 - \beta_2\, I_2 - \theta_{21}\; I_1\right] \qquad [\text{ VI-40 }]$$

and the frequency determining equations -- *i.e.* Equation [VI-39] -- reduce to

$$\omega_1 + \dot{\varphi}_1 = \Omega_1 + \sigma_1 - \rho_1\, I_1 - \tau_{12}\, I_2$$
$$\omega_2 + \dot{\varphi}_2 = \Omega_2 + \sigma_2 - \rho_2\, I_2 - \tau_{21}\, I_1 w \qquad [\text{ VI-41 }]$$

Multiplying the amplitude equations by $\left|\mathcal{E}_s\right|^2 / \left(\hbar^2 \gamma_a \gamma_b\right)$ we obtain the equations of motion for the dimensionless intensities

$$\dot{I}_1 = 2\, I_1\left[a_1 - \beta_1\, I_1 - \theta_{12}\; I_2\right]$$
$$\dot{I}_2 = 2\, I_2\left[a_2 - \beta_2\, I_2 - \theta_{21}\; I_1\right] \qquad [\text{ VI-42 }]$$

STABILITY OF POSSIBLE STEADY STATE SOLUTIONS:

Stability criterion for stationary solutions: If $I_1 = I_1^{(s)} + \varepsilon$ and $I_2 = I_2^{(s)} + \varepsilon_2$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $t \to \infty$.

Solution 1: $I_1^{(s)} = 0$ and $I_2^{(s)} = a_2/\beta_2$ so that

$$\begin{aligned} \varepsilon_1 &= 2[\varepsilon_1]\left[a_1 - \beta_1 \varepsilon_1 - \theta_{12}\left(a_2/\beta_2 + \varepsilon_2\right)\right] \\ \varepsilon_2 &= 2[a_2/\beta_2 + \varepsilon_2]\left[a_2 - \beta_2\left(a_2/\beta_2 + \varepsilon_2\right) - \theta_{21}\varepsilon_1\right] \end{aligned} \qquad \text{[VI-43a]}$$

or

$$\begin{aligned} \dot{\varepsilon}_1 &= 2\,\varepsilon_1\left[a_1 - \theta_{12}\, a_2/\beta_2\right] + O\left(\varepsilon^2\right) \\ \dot{\varepsilon}_2 &= -2\left(a_2/\beta_2\right)\left[\beta_2\,\varepsilon_2 + \theta_{21}\,\varepsilon_1\right] + O\left(\varepsilon^2\right) \end{aligned} \qquad \text{[VI-43b]}$$

Stability requires that $a_1' \equiv a_1 - \theta_{12}\, a_2/\beta_2$ remain negative. If $a_1' < 0$ but $a_1 > 0$ one says that I_2 inhibits the oscillation of I_1 (**mode inhibition**). If a_1 becomes large to overcome the inhibiting effect of I_2 , the solution becomes unstable and I_1 builds up.

Solution 2: $I_1^{(s)} \neq 0$ and $I_2^{(s)} \neq 0$ so that

$$\begin{aligned} a_1 &= \beta_1\, I_1^{(s)} + \theta_{12}\, I_2^{(s)} \\ a_2 &= \beta_2\, I_2^{(s)} + \theta_{21}\, I_1^{(s)} \end{aligned} \qquad \text{[VI-44]}$$

which has the solution

$$\begin{aligned} I_1^{(s)} &= \frac{\left(a_1/\beta_1\right) - \left(\theta_{21}/\beta_1\right)\left(a_2/\beta_2\right)}{1-C} = \frac{a_1'/\beta_1}{1-C} \\ I_2^{(s)} &= \frac{\left(a_2/\beta_2\right) - \left(\theta_{12}/\beta_2\right)\left(a_1/\beta_1\right)}{1-C} = \frac{a_2'/\beta_2}{1-C} \end{aligned} \qquad \text{[VI-45]}$$

where the ***coupling constant*** $C = \dfrac{\theta_{12}\,\theta_{21}}{\beta_1\,\beta_2}$. Substituting this solution into Equation [VI-42] we see that

$$\begin{aligned} \dot{\varepsilon}_1 &= -2\, I_1^{(s)}\left[\beta_1\,\varepsilon_1 + \theta_{12}\,\varepsilon_2\right] + O\left(\varepsilon^2\right) \\ \dot{\varepsilon}_2 &= -2\, I_2^{(s)}\left[\beta_2\,\varepsilon_2 + \theta_{21}\,\varepsilon_1\right] + O\left(\varepsilon^2\right) \end{aligned} \qquad \text{[VI-46a]}$$

In matrix form

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} = \frac{-2}{1-C}\begin{bmatrix} a_1' & a_1'\left(\theta_{12}/\beta_1\right) \\ a_2'\left(\theta_{21}/\beta_2\right) & a_2' \end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} = \mathcal{M}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \qquad \text{[VI-46b]}$$

The solutions will be stable if the eigenvalues of this equation are both negative.

$$\mathcal{M}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} = \lambda\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix} \qquad \mathrm{Det}\left(\mathcal{M} - \lambda\, I\right) = 0 \qquad \text{[VI-47]}$$

$$\lambda_{\left(\frac{1}{2}\right)} = -\frac{a_1' + a_2'}{1-C} \pm \sqrt{\left(\frac{a_1' + a_2'}{1-C}\right)^2 - 4\left(\frac{a_1' a_2'}{1-C}\right)} \qquad \text{[VI-48]}$$

Case 1: **Weak coupling** where $C < 1$ and $a_1' > 0$ and $a_2' > 0$: **Stable** since both eigenvalues negative

Case 2: **Strong coupling** $C > 1$ and $a_1' < 0$ and $a_2' < 0$: **Unstable** since one eigenvalue is positive