II. CANONICAL QUANTIZATION OF ELECTRODYNAMICS:

With the foregoing preparation, we are now in a position to apply the classical analogy or canonical quantization program to achieve the second quantization of the electromagnetic field. As our starting point and for reference, we, once again, set forth the vacuum or microscopic Maxwell's equations in the time domain:

\[
\begin{align*}
\nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \quad [\text{II-1a}] \\
\nabla \times \mathbf{B}(\mathbf{r}, t) &= \mu_0 \mathbf{J}(\mathbf{r}, t) + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) \quad [\text{II-1b}] \\
\n\nabla \cdot \mathbf{E}(\mathbf{r}, t) &= \rho(\mathbf{r}, t) / \epsilon_0 \quad [\text{II-1c}] \\
\n\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0 \quad [\text{II-1d}] 
\end{align*}
\]

The canonical formulation of classical electrodynamics (Jeans' Theorem) is most conveniently achieved in terms of the (magnetic) vector potential in the time domain -- viz.

\[
\begin{align*}
\mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) \quad [\text{II-2a}] \\
\mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) - \nabla \phi(\mathbf{r}, t) \quad [\text{II-2b}] 
\end{align*}
\]

so that

\[
\begin{align*}
\nabla \cdot \mathbf{A}(\mathbf{r}, t) - \nabla^2 \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A}(\mathbf{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \phi(\mathbf{r}, t) &= \mu_0 \mathbf{J}(\mathbf{r}, t) \quad [\text{II-3a}] 
\end{align*}
\]

---

5 In common usage, the process of treating the coordinates \(q_i\) and \(p_i\) as quantized variables is called \textit{first quantization}. \textit{Second quantization} is the process of quantizing \textit{fields} -- say, \(\mathbf{A}(\mathbf{r}, t)\) -- which have an infinite number of degrees of freedom.
THE INTERACTION OF RADIATION AND MATTER: QUANTUM THEORY

\[-\varepsilon_0 \nabla \cdot \frac{\partial}{\partial t} \vec{A}(\vec{r}, t) - \varepsilon_0 \nabla^2 \phi(\vec{r}, t) = \rho(\vec{r}, t)\]

[II-3b]

In QED (Quantum Electrodynamics) it is convenient and traditional to make use of the Coulomb gauge -- *i.e.* \( \vec{V} \cdot \vec{A}(\vec{r}, t) = 0 \) -- so that

\[\nabla^2 \vec{A}(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A}(\vec{r}, t) = -\mu_0 \vec{J}_T(\vec{r}, t)\]

[II-4a]

\[\nabla^2 \phi(\vec{r}, t) = -\rho(\vec{r}, t)/\varepsilon_0\]

[II-4b]

where \( \vec{J}_T(\vec{r}, t) = \vec{J}(\vec{r}, t) - \vec{J}_L(\vec{r}, t) = \vec{J}(\vec{r}, t) - \varepsilon_0 \frac{\partial}{\partial t} \phi(\vec{r}, t) \) is the so-called transverse current density. Since \( \vec{A}(\vec{r}, t) \) is completely determined by the transverse current density in the Coulomb gauge, electromagnetic problems become in a sense separable -- *i.e.*

**The transverse field problem:**

\[\vec{V} \cdot \vec{E}_T(\vec{r}, t) = 0\]

\[\vec{V} \times \vec{E}_T(\vec{r}, t) = -\mu_0 \frac{\partial}{\partial t} \vec{H}(\vec{r}, t)\]

[II-5a]

\[\vec{V} \times \vec{H}_T(\vec{r}, t) = \vec{J}_T(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}_T(\vec{r}, t)\]

**The longitudinal field problem:**

\[\vec{V} \cdot \vec{E}_L(\vec{r}, t) = \rho(\vec{r}, t)/\varepsilon_0\]

\[\vec{J}_L(\vec{r}, t) = -\varepsilon_0 \frac{\partial}{\partial t} \vec{E}_L(\vec{r}, t)\]

[II-5b]
We turn now explicitly to a treatment of the free electromagnetic field -- formally the case of \( \mathbf{J}_t (\mathbf{r}, t) \equiv 0 \) wherein
\[
\nabla^2 \tilde{\mathbf{A}} (\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tilde{\mathbf{A}} (\mathbf{r}, t) = 0 \tag{II-6}
\]

We look for solutions in the form
\[
\tilde{\mathbf{A}} (\mathbf{r}, t) = \frac{1}{2\sqrt{\varepsilon_0}} \sum_s \{ \alpha_s (t) \tilde{\mathbf{u}}_s (\mathbf{r}) + \text{c.c.} \} \tag{II-7}
\]

Substituting into Equation [II-6] we obtain
\[
\sum_s \left\{ \alpha_s (t) \tilde{\mathbf{u}}_s (\mathbf{r}) \left[ \nabla^2 \tilde{\mathbf{u}}_s (\mathbf{r}) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tilde{\mathbf{u}}_s (\mathbf{r}) \right] + \text{c.c.} \right\} = 0 \tag{II-8}
\]

Thus we may apply separation of variables techniques and the original problem is divided into two new and distinct problems -- viz. solutions of the following set of equations:
\[
\nabla^2 \tilde{\mathbf{u}}_s (\mathbf{r}) + \frac{\omega^2_s}{c^2} \tilde{\mathbf{u}}_s (\mathbf{r}) = 0 \tag{II-9}
\]
\[
\dot{\alpha}_s (t) + \omega^2_s \alpha_s (t) = 0 \tag{II-10}
\]
where the \( \omega_s \)'s are separation constants. The spatial equations allow us to treat the boundary value problem of the cavity or defining field space in whatever detail that might seem appropriate in a particular case. 6 But the bottom line is that the boundary condition

---

6 The \( \tilde{\mathbf{u}}_s (\mathbf{r}) \)'s do not form a true complete set of solutions since no longitudinal vector field can be expanded in terms of such divergentless functions.
THE INTERACTION OF RADIATION AND MATTER: QUANTUM THEORY

taken together with the characteristics of the normal modes or eigenfunctions determine the
\( \omega_s \)'s which may, in turn, be identified as the eigenfrequencies of the normal modes.

Thus, we may now write

\[
\dot{\mathbf{E}}_\text{T}(\mathbf{r},t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r},t) = -\frac{1}{2\varepsilon_0} \sum_s \{ \alpha_s(t) \mathbf{u}_s(\mathbf{r}) + \text{c.c.} \}
\]

\[
= -\frac{1}{2\varepsilon_0} \sum_s \left\{ i \omega_s \alpha_s(t) \mathbf{u}_s(\mathbf{r}) + \text{c.c.} \right\} \tag{II-11}
\]

and

\[
\dot{\mathbf{H}}_\text{T}(\mathbf{r},t) = \frac{1}{\mu_0} \nabla \times \mathbf{A}(\mathbf{r},t) = \frac{c}{2\mu_0} \sum_s \left\{ \alpha_s(t) \nabla \times \mathbf{u}_s(\mathbf{r}) + \text{c.c.} \right\}. \tag{II-12}
\]

The crucial step required in establishing Jeans' Theorem is the expansion of the
instantaneous value of the stored electromagnetic energy in terms of the cavity modes --
viz.

\[
\langle W(t) \rangle = \frac{1}{2V} \iiint \left[ \varepsilon_0 |\mathbf{E}(\mathbf{r},t)|^2 + \mu_0 |\mathbf{H}(\mathbf{r},t)|^2 \right] dV \tag{II-13a}
\]

which, in light of Equations \( \text{[II-11]} \) and \( \text{[II-12]} \), becomes

\[
\langle W(t) \rangle = \frac{1}{4V} \sum_{s,s'} \iiint \left[ \omega_s \omega_{s'} \mathbf{u}_s \cdot \mathbf{u}_{s'}^* + c^2 \nabla \times \mathbf{u}_s \cdot \nabla \times \mathbf{u}_{s'}^* \right] \alpha_s(t) \alpha_{s'}^*(t) dV \tag{II-13b}
\]
THE INTERACTION OF RADIATION AND MATTER: QUANTUM THEORY

To proceed we need the value of the integral \[ \int \int \int_{\text{cavity}} \left[ \vec{\nabla} \times \vec{u}_s \cdot \vec{\nabla} \times \vec{u}^*_s \right] \, dV. \] Using a well-known vector identity,\(^7\) we obtain

\[
\int \int \int_{\text{cavity}} \left[ \vec{\nabla} \times \vec{u}_s \cdot \vec{\nabla} \times \vec{u}^*_s \right] \, dV = \int \int \int_{\text{cavity}} \vec{u}_s \times \left( \vec{\nabla} \times \vec{u}^*_s \right) \cdot d\vec{S} + \int \int \int_{\text{cavity}} \vec{u}_s \cdot \vec{\nabla} \times \left( \vec{\nabla} \times \vec{u}^*_s \right) \, dV \tag{II-14a}
\]

By boundary value arguments we may easily show that the first term on the RHS of this equation vanishes and by using an even more familiar (famous) vector identity\(^8\) for a divergenceless field, we obtain

\[
\int \int \int_{\text{cavity}} \left[ \vec{\nabla} \times \vec{u}_s \cdot \vec{\nabla} \times \vec{u}^*_s \right] \, dV = -\int \int \int_{\text{cavity}} \vec{u}_s \cdot \nabla^2 \vec{u}^*_s \, dV = \frac{\omega^2_s}{c^2} \int \int \int_{\text{cavity}} \vec{u}_s \cdot \vec{u}^*_s \, dV \tag{II-14b}
\]

Therefore, Equation [ II-13b ] becomes

\[
\langle W(t) \rangle = \frac{1}{4V} \sum_{s,s'} \left[ \omega_s \omega_{s'} + \omega^2_{s'} \right] \alpha_{s}(t) \alpha^*_{s'}(t) \int \int \int_{\text{cavity}} \vec{u}_s \cdot \vec{u}^*_s \, dV \tag{II-15}
\]

From Equation [ II-14b ] we may also write

\[\text{Namely, that } \vec{\nabla} \cdot \left( \vec{X} \times \vec{\nabla} \times \vec{Y} \right) = \left( \vec{\nabla} \times \vec{X} \right) \cdot \left( \vec{\nabla} \times \vec{Y} \right) - \vec{X} \cdot \left( \vec{\nabla} \times \vec{\nabla} \times \vec{Y} \right)\]

\[\text{Namely, that } \nabla \left[ \vec{X} \times \left( \vec{\nabla} \times \vec{Y} \right) \right] = \left( \vec{\nabla} \times \vec{X} \right) \cdot \left( \vec{\nabla} \times \vec{Y} \right) - \vec{X} \cdot \vec{\nabla} \times \left( \vec{\nabla} \times \vec{Y} \right) = \left( \vec{\nabla} \times \vec{X} \right) \cdot \left( \vec{\nabla} \times \vec{Y} \right) - \vec{X} \cdot \vec{\nabla} \left( \vec{\nabla} \cdot \vec{Y} \right) + \vec{X} \cdot \nabla^2 \vec{Y} \]
THE INTERACTION OF RADIATION AND MATTER: QUANTUM THEORY

\[ \iiint \tilde{\nabla} \times \left[ \tilde{\nabla} \times \tilde{\nabla} \times \tilde{u}_s \right] dV = \frac{1}{c^2} \left( \omega_s^2 - \omega_s^* \right) \iiint \tilde{u}_s \cdot \tilde{u}_s^* dV \] II-16a

and, again, by boundary value arguments we may easily show that the resultant surface integral on the left \( \omega_s^2 - \omega_s^* \) \( \iiint \tilde{u}_s \cdot \tilde{u}_s^* dV = 0 \) II-16b

Therefore, we may take

\[ \iiint \tilde{u}_s \cdot \tilde{u}_s^* dV = \delta_{ss'} \] II-17

so that Equation [ II-15 ] becomes

\[ \langle W(n) \rangle = \frac{1}{2} \sum_s \omega_s^2 \alpha_s \alpha_s^* \] II-18

which when compared to Equation [ I-6 ]\(^9\) is effectively the content of Jean's Theorem with

\[ \sqrt{\frac{\omega_s}{2}} \alpha_s \Rightarrow a = \sqrt{\frac{m}{2 \omega}} \left[ \omega q + i \frac{p}{m} \right] \] II-19

\[ \sqrt{\frac{\omega_s}{2}} \alpha_s^* \Rightarrow a' = \sqrt{\frac{m}{2 \omega}} \left[ \omega q - i \frac{p}{m} \right] \]

\(^9\) That is \( \mathcal{H} = \omega a^\dagger a \)
THE INTERACTION OF RADIATION AND MATTER: QUANTUM THEORY

To accomplish the canonical quantization program, field variables are expressed as field operators by making the identification

\[
\sqrt{\frac{\omega_s}{2}} \alpha_s \Rightarrow \sqrt{\frac{m}{2 \omega}} [\omega q + i \frac{p}{m}] = \sqrt{\hbar} a
\]

\[
\sqrt{\frac{\omega_s}{2}} \alpha_s^* \Rightarrow \sqrt{\frac{m}{2 \hbar \omega}} [\omega q - i \frac{p}{m}] = \sqrt{\hbar} a^\dagger
\]

which leads to the following set of field operators:

\[
\mathcal{H} = \sum_s \frac{\hbar \omega_s}{2} \left\{ a_s a_s^\dagger + a_s^\dagger a_s \right\}
\]

\[
\tilde{A}(\vec{r},t) = \sum_s \sqrt{\frac{\hbar}{2 \omega_s \varepsilon_0}} \left\{ a_s(t) \tilde{\mathbf{u}}_s(\vec{r}) + a_s^\dagger(t) \tilde{\mathbf{u}}_s^* \right\}
\]

\[
\tilde{E}_t(\vec{r},t) = \sum_s \sqrt{\frac{\hbar \omega_s}{2 \varepsilon_0}} \left\{ a_s(t) \tilde{\mathbf{u}}_s(\vec{r}) - a_s^\dagger(t) \tilde{\mathbf{u}}_s^* \right\}
\]

\[
\tilde{H}_t(\vec{r},t) = c \sum_s \sqrt{\frac{\hbar}{2 \omega_s \mu_0}} \left\{ a_s(t) \hat{\nabla} \times \tilde{\mathbf{u}}_s(\vec{r}) + a_s^\dagger(t) \hat{\nabla} \times \tilde{\mathbf{u}}_s^* \right\}
\]

The Plane Wave Expansion of the Electromagnetic Field Hamiltonian:

To be definite, we may write an explicit plane wave representation for the field as

\[
\tilde{A}(\vec{r},t) = \sum_s \sum_{\alpha=1}^2 \sqrt{\frac{\hbar}{2 \omega_s \varepsilon_0 \mu_0}} \tilde{e}_{i,\alpha} \left\{ a_{s,\alpha} \exp\left[i(\vec{k}_s \cdot \vec{r} - \omega_s t)\right] + a_{s,\alpha}^\dagger \exp\left[-i(\vec{k}_s \cdot \vec{r} - \omega_s t)\right] \right\}
\]

[ II-20 ]
or

\[ \tilde{A}(\mathbf{r}, t) = \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar}{2 \omega_s \varepsilon_0}} \left\{ a_{s\sigma}(t) \tilde{u}_{s\sigma}(\mathbf{r}) + a_{s\sigma}^\dagger(t) \tilde{u}_{s\sigma}^\dagger(\mathbf{r}) \right\} \]  

[II-22b]

where

\[ \tilde{u}_{s\sigma}(\mathbf{r}) = \frac{1}{\sqrt{V}} \hat{\epsilon}_{s\sigma} \exp\left[i \mathbf{k}_s \cdot \mathbf{r}\right] \quad \text{and} \quad \tilde{u}_{s\sigma}^\dagger(\mathbf{r}) = \frac{1}{\sqrt{V}} \hat{\epsilon}_{s\sigma} \exp\left[-i \mathbf{k}_s \cdot \mathbf{r}\right] \]  

[II-23a]

\[ a_{s\sigma}(t) = a_{s\sigma}(0) \exp\left[-i \omega_s t\right] \quad \text{and} \quad a_{s\sigma}^\dagger(t) = a_{s\sigma}^\dagger(0) \exp\left[i \omega_s t\right] \]  

[II-23b]

Of course, \( k_s = |\mathbf{k}_s| = \omega_s / c \) in all of these expansions. Further the electric field expansion is given by

\[ \tilde{E}(\mathbf{r}, t) = i \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar \omega_s}{2 \varepsilon_0}} \left\{ a_{s\sigma}(t) \tilde{u}_{s\sigma}(\mathbf{r}) - a_{s\sigma}^\dagger(t) \tilde{u}_{s\sigma}^\dagger(\mathbf{r}) \right\} \]  

[II-24a]

\[ = i \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar \omega_s}{2 \varepsilon_0}} \hat{E}_s \left\{ a_{s\sigma}(t) \exp\left[i \mathbf{k}_s \cdot \mathbf{r} - i \omega_s t\right] - a_{s\sigma}^\dagger(t) \exp\left[i \mathbf{k}_s \cdot \mathbf{r} - i \omega_s t\right]\right\} \]

where \( \hat{E}_s = \sqrt{\frac{\hbar \omega_s}{2 \varepsilon_0 V}} \) and the magnetic field expansion by

\[ \tilde{H}(\mathbf{r}, t) = \frac{i}{c \mu_0} \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar \omega_s}{2 \varepsilon_0}} \left\{ a_{s\sigma}(t) \left[ \mathbf{k}_s \times \tilde{u}_{s\sigma}(\mathbf{r}) \right] - a_{s\sigma}^\dagger(t) \left[ \mathbf{k}_s \times \tilde{u}_{s\sigma}^\dagger(\mathbf{r}) \right]\right\} \]  

[II-24b]

\[ \tilde{E}(\mathbf{r}, t) = \tilde{E}^{(+)}(\mathbf{r}, t) + \tilde{E}^{(-)}(\mathbf{r}, t) \]  

where

\[ \tilde{E}^{(+)}(\mathbf{r}, t) = i \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar \omega_s}{2 \varepsilon_0}} \hat{E}_s a_{s\sigma}(t) \exp\left[i \mathbf{k}_s \cdot \mathbf{r} - i \omega_s t\right] \quad \text{and} \quad \tilde{E}^{(-)}(\mathbf{r}, t) = -i \sum_{s=1}^{2} \sum_{\sigma=1}^{\frac{\hbar \omega_s}{2 \varepsilon_0}} \hat{E}_s a_{s\sigma}^\dagger(t) \exp\left[i \mathbf{k}_s \cdot \mathbf{r} - i \omega_s t\right] \]
The interaction of radiation and matter: quantum theory

In light of Equations [II-18] and [II-21], the Hamiltonian of the radiation field is

\[ \mathcal{H}^{\text{rad}} = \frac{1}{2} \sum_{\{l\}} \sum_{\alpha=1}^{2} \hbar \omega_{\{l\}} \left\{ a_{\{l\} \alpha} a_{\{l\} \alpha}^{\dagger} + a_{\{l\} \alpha}^{\dagger} a_{\{l\} \alpha} \right\} = \sum_{\{l\}} \sum_{\alpha=1}^{2} \hbar \omega_{\{l\}} \left\{ \mathcal{N}_{\{l\} \alpha} + 1/2 \right\} \]  

where

\[ \mathcal{N}_{\{l\} \alpha} = \frac{2 \pi}{L} \left[ l_x \hat{x} + l_y \hat{y} + l_z \hat{z} \right] \]  

The electromagnetic momentum (Poynting vector divided by \( c^2 \)) is given classical by

\[ \mathcal{M} = \frac{1}{c^2} \iiint_{\text{cavity}} \mathbf{E} \times \mathbf{H} \, dV \]  

and in terms of the second quantization operators it becomes

\[ \mathcal{M} = \frac{1}{2} \sum_{\{l\}} \sum_{\alpha=1}^{2} \hbar \mathbf{k}_{\{l\}} \left\{ a_{\{l\} \alpha} a_{\{l\} \alpha}^{\dagger} + a_{\{l\} \alpha}^{\dagger} a_{\{l\} \alpha} \right\} \]  

\[ = \sum_{\{l\}} \sum_{\alpha=1}^{2} \hbar \mathbf{k}_{\{l\}} \mathcal{N} \]
Thus, the Fock or number states are eigenstates of both the energy and the momentum of the field.