III. REPRESENTATIONS OF PHOTON STATES

1. Fock or "Number" States: .[11]

As we have seen, the Fock or *number* states

$$\left|\left\{n_{\vec{\mathbf{k}}}\right\}\right\rangle \equiv \prod_{s\,\sigma}\left|n_{\vec{\mathbf{k}}_s\,\sigma}\right\rangle \qquad \text{[III-1]}$$

are complete set eigenstates of an important group of commuting observables -- *viz*. $\mathcal{H}_{rad}$, $\mathcal{N}$ and $\vec{\mathcal{M}}$.

Reprise of Characteristics and Properties of Fock States:

a. **The expectation value of the number operator and the *fractional uncertainty* associated with a single Fock state**:

$$\left\langle n\right|\mathcal{N}\left|n\right\rangle = n \qquad \text{[III-2a]}$$

$$\Delta n = [\text{ "uncertainty" }] = \sqrt{\left\{\left\langle n\right|\mathcal{N}^{2}\left|n\right\rangle - \left\langle n\right|\mathcal{N}\left|n\right\rangle^{2}\right\}} = 0 \qquad \text{[III-2b]}$$

b. **Expectation value of the fields associated with a single mode**:

For one mode Equations [II-24a] and [II-24b] reduce to

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right) = i\,\hat{\mathbf{e}}\,\mathcal{E}\left\{a\,\exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right] - a^{\dagger}\exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right]\right\} \qquad \text{[III-3a]}$$

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right) = i\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathcal{E}\left[\hat{\mathbf{k}}\times\hat{\mathbf{e}}\right]\left\{a\left(t\right)\exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right] - a^{\dagger}\left(t\right)\exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right]\right\} \qquad \text{[III-3b]}$$

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[11] In what follows, for simplicity we drop the $\vec{\mathbf{k}}$ subscripts on the operators and state vectors with the obvious meaning that $\left|\left\{n_{\vec{\mathbf{k}}}\right\}\right\rangle \quad \left|n\right\rangle$, $a_{\vec{\mathbf{k}}} \quad a$, *etc...*

where $\mathcal{E} = \sqrt{\dfrac{\hbar\omega}{2\varepsilon_0 V}}$

$$\begin{aligned} \langle n | \vec{\mathbf{E}} | n \rangle &= 0 \\ \langle n | \vec{\mathbf{H}} | n \rangle &= 0 \end{aligned} \qquad \text{[III-4a]}$$

$$\begin{aligned} \Delta\mathbf{E} &= \sqrt{\left\{ \langle n | \vec{\mathbf{E}} \;\; \vec{\mathbf{E}} | n \rangle - \langle n | \vec{\mathbf{E}} | n \rangle^2 \right\}} = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(n + \tfrac{1}{2} \right)^{1/2} = \sqrt{2}\, \mathcal{E} \left(n + \tfrac{1}{2} \right)^{1/2} \\ \Delta\mathbf{H} &= \sqrt{\left\{ \langle n | \vec{\mathbf{H}} \;\; \vec{\mathbf{H}} | n \rangle - \langle n | \vec{\mathbf{H}} | n \rangle^2 \right\}} = \sqrt{\frac{\hbar\omega}{\mu_0 V}} \left(n + \tfrac{1}{2} \right)^{1/2} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \sqrt{2}\, \mathcal{E} \left(n + \tfrac{1}{2} \right)^{1/2} \\ \Delta\mathbf{E}\; \Delta\mathbf{H} &= c\, \frac{\hbar\omega}{V} \left(n + \tfrac{1}{2} \right) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\; 2\, \mathcal{E}^2 \left(n + \tfrac{1}{2} \right) \end{aligned} \qquad \text{[III-4b]}$$

c. **Phase of field associated with single mode**:

To obtain something analogous to the classical theory we would like to separate the creation and destruction operators (and, thus, the electric and magnetic field operators) into a product of amplitude and phase operators. Following Susskind and Glogower,[12] we define a ***phase operator***, Φ such that

$$\begin{aligned} a &\equiv \left(\mathcal{N} + 1 \right)^{1/2} \exp\left(i\, \Phi \right) \\ a^\dagger &\equiv \exp\left(-i\, \Phi \right) \left(\mathcal{N} + 1 \right)^{1/2} \end{aligned} \qquad \text{[III-5]}$$

Defined in this way, the basic properties of the phase operator may be evaluated from known properties of the creation, destruction and number operators. Inverting, we obtain

12 Susskind, L. and Glogower, J., *Physics*, **1**, 49 (1964)

$$\exp(i\,\Phi) \equiv \left(\mathcal{N}+1\right)^{-1/2} a$$
$$\exp(-i\,\Phi) \equiv a^{\dagger}\left(\mathcal{N}+1\right)^{-1/2} \qquad \text{[III-6]}$$

and since $a\,a^{\dagger} = \mathcal{N}+1$, it follows that

$$\exp(i\,\Phi)\,\exp(-i\,\Phi) = 1 \qquad \text{[III-7]}$$

but only in this order! Operating on number states with the phase operators, we obtain from Equation [I-26]

$$\exp(i\,\Phi)\,|n\rangle = \left(\mathcal{N}+1\right)^{-1/2} a\,|n\rangle = \left(\mathcal{N}+1\right)^{-1/2}(n)^{1/2}\,|n-1\rangle = |n-1\rangle$$
$$\exp(-i\,\Phi)\,|n\rangle = a^{\dagger}\left(\mathcal{N}+1\right)^{-1/2}|n\rangle = a^{\dagger}\,(n+1)^{-1/2}\,|n\rangle = |n+1\rangle \qquad \text{[III-8]}$$

Consequently, the **only nonvanishing matrix elements** of the phase operator are

$$\langle n-1|\exp(i\,\Phi)\,|n\rangle = 1$$
$$\langle n+1|\exp(-i\,\Phi)\,|n\rangle = 1 \qquad \text{[III-9]}$$

The phase operators defined by Equation [III-36] do have the felicitous or *classically analogous* property of revealing ***magnitude independent*** information, but unfortunately they are nonHermitian operators -- *i.e.*

$$\langle n-1|\exp(i\,\Phi)\,|n\rangle \neq \langle n|\exp(i\,\Phi)\,|n-1\rangle$$

-- and, hence, **cannot represent observables**. However, they may be **paired** into operators that are observables -- *viz.*

$$\cos\Phi = \frac{1}{2}\left\{ \exp(i\,\Phi) + \exp(-i\,\Phi)\right\}$$
$$\sin\Phi = \frac{1}{2\,i}\left\{ \exp(i\,\Phi) - \exp(-i\,\Phi)\right\} \qquad [\text{ III-10 }]$$

which have the following nonvanishing matrix elements:

$$\langle n-1|\cos\Phi|n\rangle = \langle n|\cos\Phi|n-1\rangle = \frac{1}{2}$$
$$\langle n-1|\sin\Phi|n\rangle = -\langle n|\sin\Phi|n-1\rangle = \frac{1}{2\,i} \qquad [\text{ III-11 }]$$

These *nearly commuting* operators[13] may be adopted as the quantum mechanical operators which represent (as we will demonstrate anon) the observable phase properties of the electromagnetic field.

For the Fock state:

$$\langle n|\cos\Phi|n\rangle = \langle n|\sin\Phi|n\rangle = 0 \qquad [\text{ III-12a }]$$

$$\Delta\cos\Phi = \Delta\sin\Phi = \sqrt{\left\{\langle n|\cos^2\Phi|n\rangle - \langle n|\cos\Phi|n\rangle^2\right\}} = \sqrt{1/2} \qquad [\text{ III-12b }]$$

$$\Delta\cos\Phi\;\Delta\sin\Phi = 1/2 \qquad [\text{ III-12c }]$$

c. **The coordinate or Schrödinger representation of state:**

Recall from Equations [I-10a] and [I-31] that

13 Also, it may be easily established that the matrix elements of their commutator are given by

$$\langle n|[\cos\Phi,\sin\Phi]|n'\rangle = \frac{i}{2}\,\delta_{nn'}\,\delta_{n0}$$

$$\langle q|n\rangle = \frac{1}{\sqrt{n!}}\left[\sqrt{\frac{m}{2\hbar\omega}}\right]^n \left[\omega q - \frac{\hbar}{m}\frac{d}{dq}\right]^n \langle q|0\rangle$$

$$= \sqrt{\frac{1}{2^n\, n!}\sqrt{\frac{\omega}{\pi\hbar}}}\; H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, q\right)\exp\left[-\frac{m\omega}{2\hbar}q^2\right] \qquad \text{[III-13]}$$

Therefore, the probability $P(q)$ of eigenvalues q for a given Fock state $|n\rangle$ is give by

$$P(q) = \left|\langle q|n\rangle\right| = \frac{1}{2^n\, n!}\sqrt{\frac{\omega}{\pi\hbar}}\; H_n^2\left(\sqrt{\frac{m\omega}{\hbar}}\, q\right)\exp\left[-\frac{m\omega}{\hbar}q^2\right] \qquad \text{[III-14]}$$

d. **Approximate "localization" of a photon:** [14]

Of course a plane wave is distributed or "de-localized" in both time and space. Defining the "wave function for a photon" is a task fraught with danger,[15] but the simpler task of defining a wave function approximately localized at a given instant is relatively straight forward -- *viz*.

$$\left|\psi\left(\vec{\mathbf{r}}_{\mathbf{0}}\right)\right\rangle_{\vec{\mathbf{k}}_0 |\Delta\vec{\mathbf{k}}|^2} = C\sum_{\vec{\mathbf{k}}}\exp\left[\frac{\left|\vec{\mathbf{k}}-\vec{\mathbf{k}}_0\right|^2}{2\left|\Delta\vec{\mathbf{k}}\right|^2}\right]\exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}_{\mathbf{0}}\right]\left|0,0,0,\ldots n_{\vec{\mathbf{k}}}=1,\ldots,0,0,0\right\rangle \qquad \text{[III-14]}$$

2. **<u>Photon States of *Well-defined* Phase</u>:**

Consider the state defined by

$$|\phi\rangle \equiv \lim_{s\to\infty}(s+1)^{-1/2}\sum_{n=0}^{s}\exp\left[i\,n\,\phi\right]|n\rangle \qquad \text{[III-15]}$$

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[14] See Section 10.4.2 in Leonard Mandel and Emil Wolf, *Optical Coherence and Quantum Optics*, Cambridge Press (1995), ISBN 0-521-417112.

[15] See Section 1.5.4 in Marlan O. Scully and M. Suhail Zubairy, Quantum Optics, Cambridge Press (1997), ISBN 0-521-43458.

Clearly, $\langle \phi | \phi \rangle = 1$ given the orthonormal properties of the number states. Essential question: Is this state an eigenstate of the phase operators? To answer the question we need to consider the following ***potential eigenvalue equation*:**

$$\cos\Phi\,|\phi\rangle = \frac{1}{2}\lim_{s\to\infty}(s+1)^{-1/2}\left\{\sum_{n=0}^{s}\exp[i\,n\,\phi]\,\exp[i\,\Phi]\,|n\rangle + \sum_{n=0}^{s}\exp[i n\,\phi]\,\exp[-i\,\Phi]\,|n\rangle\right\} \qquad [\text{ III-16a }]$$

Using Equations [III-10] and [III-10], we obtain

$$\begin{aligned}
\cos\Phi\,|\phi\rangle &= \frac{1}{2}\lim_{s\to\infty}(s+1)^{-1/2}\left\{\sum_{n=1}^{s}\exp[i n\,\phi]\,|n-1\rangle + \sum_{n=0}^{s}\exp[i\,n\,\phi]\,|n+1\rangle\right\} \\
&= \frac{1}{2}\lim_{s\to\infty}(s+1)^{-1/2}\left\{\exp(i\phi)\sum_{\nu=0}^{s-1}\exp[i\,\nu\,\phi]\,|\nu\rangle + \exp[-i\,\phi]\sum_{\nu=1}^{s+1}\exp[i\,\nu\,\phi]\,|\nu\rangle\right\} \qquad [\text{ III-16b }] \\
&= \cos\phi\,|\phi\rangle \\
&\qquad + \frac{1}{2}\lim_{s\to\infty}(s+1)^{-1/2}\left\{\exp[i\,s\,\phi]\,|s+1\rangle - \exp[i\,(s+1)\,\phi]\,|s\rangle - \exp[-i\,\phi]\,|0\rangle\right\}
\end{aligned}$$

so that the state $|\phi\rangle$ fails to be a strict eigenket of $\cos\Phi$ by terms that diminish faster than $(s+1)^{-1/2}$ as $s\to\infty$. Similarly, we can see that diagonal matrix elements of $\cos\Phi$ and $\sin\Phi$ are given by

$$\langle\phi|\cos\Phi|\phi\rangle = \cos\phi\left\{1 - \lim_{s\to\infty}(s+1)^{-1}\right\} \qquad \cos\phi \qquad [\text{ III-17a }]$$

$$\langle\phi|\sin\Phi|\phi\rangle = \sin\phi\left\{1 - \lim_{s\to\infty}(s+1)^{-1}\right\} \qquad \sin\phi \qquad [\text{ III-17b }]$$

Reprise of Characteristics and Properties of Phase States:

a. **The expectation value of the number operator and the *fractional uncertainty* associated with a state of well-defined phase**:

$$\langle\phi|\mathcal{N}|\phi\rangle = \lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s} n = \lim_{s\to\infty}(s+1)^{-1}\left[\frac{s(s+1)}{2}\right] = \lim_{s\to\infty}\frac{s}{2} \qquad \text{[III-18a]}$$

$$\begin{aligned}\left[\begin{matrix}\text{fractional}\\ \text{uncertainty}\end{matrix}\right] &= \frac{\sqrt{\left\{\langle\phi|\mathcal{N}^2|\phi\rangle - \langle\phi|\mathcal{N}|\phi\rangle^2\right\}}}{\langle\phi|\mathcal{N}|\phi\rangle} \\ &= \frac{\sqrt{\left\{\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s} n^2 - \left[\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s} n\right]^2\right\}}}{\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s} n} \\ &= \frac{\sqrt{\left\{\lim_{s\to\infty}\left[\frac{1}{6}\left(2s^2+s\right) - \frac{1}{4}s^2\right]\right\}}}{\lim_{s\to\infty}\frac{s}{2}} = \frac{1}{\sqrt{3}}\end{aligned} \qquad \text{[III-18b]}$$

b. **Expectation value of the fields associated with a single mode**:

From Equation [III-3a]

$$\langle\phi|\vec{\mathbf{E}}|\phi\rangle = -2\sqrt{\frac{\hbar\,\omega}{2\,\varepsilon_0\,V}}\;\hat{\mathbf{e}}\,\sin\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega\,t + \phi\right)\lim_{s\to\infty}(s+1)^{-1}\sum_{n=0}^{s}(n+1)^{1/2} \qquad \text{[III-19]}$$

diverges as $\sqrt{s}$ for large s !

c. **Phase of field associated with single mode**:

$$\begin{aligned} \langle \phi \,|\cos\Phi\,|\phi\rangle &= \cos\phi \\ \langle \phi \,|\sin\Phi\,|\phi\rangle &= \sin\phi \end{aligned} \qquad \text{[III-20a]}$$

$$\Delta\cos\Phi = \Delta\sin\Phi = \sqrt{\left\{\langle \phi\,|\cos^2\Phi|\phi\rangle - \langle \phi\,|\cos\Phi\,|\phi\rangle^2\right\}} = 0 \qquad \text{[III-20b]}$$

d. **Probability of photon number**:

Finally, we may easily deduce the probability of finding n photons (*i.e.* the photon statistics) in a particular state of well defined phase -- *viz.*

$$P_n = |\langle n|\phi\rangle|^2 \equiv \lim_{s\to\infty} (s+1)^{-1} \qquad \text{[III-50]}$$

We see that there is a equal, but small probability of any number: this agrees with the intuition that the magnitude of the field is completely undetermined if the phase is precisely known!

3. ***Coherent* Photon States:** [16]

It would, indeed, be useful to have eigenstates of the ***destruction operator*** (electric or magnetic field) -- *viz.*

$$a_{\vec{\mathbf{k}}}\,|\alpha_{\vec{\mathbf{k}}}\rangle = \alpha_{\vec{\mathbf{k}}}\,|\alpha_{\vec{\mathbf{k}}}\rangle \qquad \text{[III-51]}$$

Reprise of Characteristics and Properties of Coherent States:

a. **The Fock state representation of the coherent state**:

[16] The coherent state is a **Harvard invention**! See R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

Since. $a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a\,a^{\dagger} = \mathcal{N}+1$, then $\langle n|\,a = \sqrt{n+1}\,\langle n+1|$ and we are able to write a **representative** of the sought state in the number state basis -- *viz*.

$$\langle n|\,a\,|\alpha\rangle = \sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\,\langle n|\alpha\rangle \qquad \text{[III-52a]}$$

or

$$\langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle = \frac{\alpha^{n}}{\sqrt{n!}}\langle 0|\alpha\rangle \qquad \text{[III-52b]}$$

Using the expansion of the identity operator, the eigenket becomes

$$|\alpha\rangle = \sum_{n}|n\rangle\langle n|\alpha\rangle = \langle 0|\alpha\rangle \sum_{n}\frac{\alpha^{n}}{\sqrt{n!}}\,|n\rangle\ . \qquad \text{[III-53]}$$

To normalize the eigenket write

$$\langle\alpha|\alpha\rangle = \langle\alpha|0\rangle\langle 0|\alpha\rangle \sum_{n}\frac{\alpha^{*n}\,\alpha^{n}}{n!} = \langle\alpha|0\rangle\langle 0|\alpha\rangle\,\exp\left[|\alpha|^{2}\right] = 1 \qquad \text{[III-54]}$$

so that $\langle\alpha|0\rangle = \langle 0|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^{2}\right]$. Finally, we see that

$$|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^{2}\right]\sum_{n}\frac{\alpha^{n}}{\sqrt{n!}}\,|n\rangle \qquad \text{[III-55]}$$

is a normalize representation of the eigenkets of the destruction operator.

b. **The expectation value of the number operator and the *fractional uncertainty* associated with a coherent state**:

$$\langle \alpha | \mathcal{N} | \alpha \rangle = |\alpha|^2 \qquad \text{[III-56a]}$$

$$\begin{bmatrix} \text{fractional} \\ \text{uncertainty} \end{bmatrix} = \frac{\sqrt{\left\{ \langle \alpha | \mathcal{N}^2 | \alpha \rangle - \langle \alpha | \mathcal{N} | \alpha \rangle^2 \right\}}}{\langle \alpha | \mathcal{N} | \alpha \rangle} = \frac{1}{|\alpha|^2} \sqrt{\left\{ \exp\left(-|\alpha|^2\right) \sum \frac{|\alpha|^{2n}}{n!} n^2 - |\alpha|^4 \right\}}$$

$$= \frac{1}{|\alpha|^2} \sqrt{\left\{ \exp\left(-|\alpha|^2\right) \sum \frac{|\alpha|^{2n}}{n!} \left[n\left(n-1\right) + n \right] - |\alpha|^4 \right\}} \qquad \text{[III-56b]}$$

$$= |\alpha|^{-1}$$

Thus, we see that the fractional uncertainty diminishes with mean photon number!

c. **Expectation value of the electric field associated with a single mode**:

From Equation [III-3a]

$$\langle \alpha | \mathbf{E} | \alpha \rangle = -2 \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \, \hat{\mathbf{e}} \, |\alpha| \sin\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \vartheta \right) \qquad \text{[III-57a]}$$

where $\alpha = |\alpha| \exp\left(i \vartheta\right)$.

$$\Delta \mathbf{E} = \sqrt{\left\{ \langle \alpha | \vec{\mathbf{E}} \cdot \vec{\mathbf{E}} | \alpha \rangle - \langle \alpha | \vec{\mathbf{E}} | \alpha \rangle^2 \right\}} = \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \text{ [17]} \qquad \text{[III-57b]}$$

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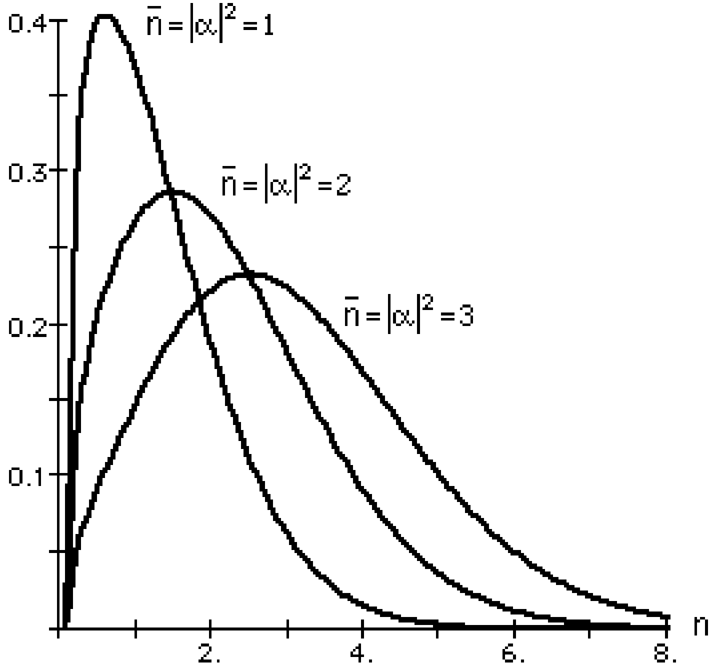
17 Similarly $\Delta \mathbf{H} = \dfrac{1}{c \, \mu_0} \sqrt{\dfrac{\hbar \omega}{2 \, \varepsilon_0 \, V}}$ for the coherent state, so that $\Delta \mathbf{E} \, \Delta \mathbf{H} = c \, \hbar \, \omega / 2 \, V$.

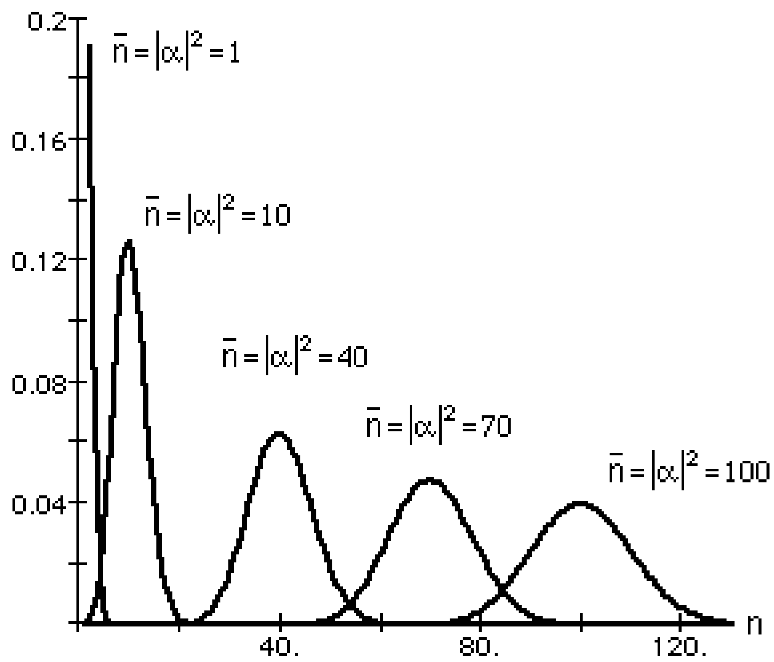
d. **Probability of photon number**:

From the representation of the coherent state given in Equation [III-55] we may easily deduce the probability of finding *n* photons (the photon statistics) in a particular coherent state is given by a **Poisson distribution** characterized by the mean value $\bar{n} = |\alpha|^2$. -- *viz*.

$$P_n = \left|\langle n | \alpha \rangle\right|^2 = \exp\left[-|\alpha|^2\right] \frac{|\alpha|^{2n}}{n!} \qquad \text{[III-58]}$$

SAMPLE POISSON DISTRIBUTIONS - COHERENT STATE PHOTON STATISTICS





e. **Phase of field associated with single mode**:

$$\begin{aligned}\langle\alpha\,|\cos\Phi|\,\alpha\rangle &= \frac{1}{2}\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n'}\sum_{n}\langle n'|\frac{\alpha^{n'}}{\sqrt{n'!}}\left[\left(\mathcal{N}+1\right)^{-1/2}a+a^{\dagger}\left(\mathcal{N}+1\right)^{-1/2}\right]\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle \\ &= \frac{1}{2}\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n}\left[\frac{\alpha^{n+1}\alpha^{n}+\alpha^{n}\alpha^{n+1}}{\sqrt{\left\{(n+1)!\,n!\right\}}}\right] \\ &= |\alpha|\cos\vartheta\,\exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_{n}\left[\frac{|\alpha|^{2n}}{n!\,\sqrt{(n+1)}}\right]\end{aligned} \qquad \text{[III-59a]}$$

Unfortunately, it is not possible to evaluate this summation analytically. However, Carruthers[18] has given an asymptotic expansion which is valid for a large mean number of photons -- *viz*.

$$\langle\alpha\,|\cos\Phi|\,\alpha\rangle = \cos\vartheta\,\left[1-\frac{1}{8|\alpha|^2}+\ldots\right] \qquad |\alpha|^2 >> 1 \qquad \text{[III-59b]}$$

f. **Coherent states as a basis:**

As we will see presently, the coherent states are very useful in describing the quantized electromagnetic field, but, alas, there is a complication -- **the coherent states are not truly orthogonal!** From Equation [III-6] we see that

$$\begin{aligned}\langle\beta|\alpha\rangle &= \exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2\right]\sum_{n}\frac{\beta^{n}\alpha^{n}}{n!} \\ &= \exp\left[-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha\beta\right]\end{aligned} \qquad \text{[III-60]}$$

so that

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[18] Carruthers, P. and Nieto, M. M., *Phys. Rev. Lett.* **14**, 387 (1965)

$$\begin{aligned}\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle &= \exp\left(-|\alpha|^2-|\beta|^2+\alpha^*\beta+\alpha\beta^*\right)\\ &= \exp\left(-(\alpha-\beta)(\alpha^*-\beta^*)\right)=\exp\left(-|\alpha-\beta|^2\right)\end{aligned} \qquad \text{[III-61]}$$

That is, t**he eigenkets are approximately orthogonal** only when $|\alpha-\beta|$ is large!

g. **The "displacement operator:"**

There are a growing and significant set of applications where it is useful to express the coherent states directly in terms of the vacuum state $|0\rangle$. If we use the number state generating rule

$$|n\rangle=\frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

-- *i.e.* Equation [I-27] -- the coherent state may be written in the form

$$|\alpha\rangle=\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n\frac{\left(\alpha\,a^\dagger\right)^n}{n!}|0\rangle=\exp\left(\alpha\,a^\dagger-\frac{1}{2}|\alpha|^2\right)|0\rangle \qquad \text{[III-62]}$$

If we make us of the Baker-Hausdorff theorem,[19] we may easily show that

[19] The Baker-Hausdorff theorem or identity may be stated as

$$\exp\{\mathcal{A}+\mathcal{B}\}=\exp\{\mathcal{A}\}\ \exp\{\mathcal{B}\}\ \exp\left\{-\tfrac{1}{2}[\mathcal{A},\mathcal{B}]\right\}$$

when $[\mathcal{A},[\mathcal{A},\mathcal{B}]]=[\mathcal{B},[\mathcal{A},\mathcal{B}]]=0$. For a proof, see, for example, Charles P. Slichter's *Principles of Magnetic Resonance*, Appendix A or William Louisell's *Radiation and Noise in Quantum Electronics*.

$$|\alpha\rangle = \mathcal{A}^{\dagger}(\alpha)\,|0\rangle = \exp\left(\alpha\, a^{\dagger} - \alpha^{*}\, a\right)|0\rangle \qquad \text{[III-63]}$$

so that $\mathcal{A}^{\dagger}(\alpha)$ may be interpreted as a *creation* operator which generates a coherent state from the vacuum. (Its adjoint operator $\mathcal{A}(\alpha) = \mathcal{A}^{\dagger}(-\alpha)$ is a *destruction* operator which destroys a state). In some treatments $\mathcal{A}^{\dagger}(\alpha)$ is described as the "displacement operator" (written $\mathcal{D}(\alpha)$)[20] and the coherent states are called the "displaced states of the vacuum." [21]

To explore this point of view (and to give some meaning to the phase of the coherent state eigenvalue), we may express $|\alpha\rangle$ in a two-dimensional, dimensionless "phase space" representation. To that end, following Equation [I-16], we write the dimensionless coordinate as

$$\theta = \left(\frac{m\omega}{\hbar}\right)^{1/2} q = a^{\dagger}\exp[i\gamma] + a\,\exp[-i\gamma] \qquad \text{[III-64a]}$$

and the dimensionless momentum as

$$\pi = \left(\frac{2}{m\,\hbar\omega}\right)^{1/2} p = a^{\dagger}\exp\left[i\left(\gamma + \pi/2\right)\right] + a\,\exp\left[-i\left(\gamma + \pi/2\right)\right] \qquad \text{[III-64b]}$$

so that

$$[\theta, \pi] = 2i\left[a, a^{\dagger}\right] = 2i \qquad \text{[III-64c]}$$

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[20] We can (or rather you will) show that $\mathcal{D}^{\dagger}(\alpha)\, a\, \mathcal{D}(\alpha) = a + \alpha$ and $\mathcal{D}^{\dagger}(\alpha)\, a^{\dagger}\, \mathcal{D}(\alpha) = a^{\dagger} + \alpha$

[21] See *Elements of Quantum Optics*, Pierre Meystre and Murray Sargent III, Spinger-Verlag (1991), ISBN 0-387-54190-X.

and since these variables are canonical [22]

$$\left\langle (\Delta\theta)^2 \right\rangle \left\langle (\Delta\pi)^2 \right\rangle \geq 1 \qquad \text{[III-64d]}$$

Since

$$\begin{aligned} a^\dagger &= \frac{1}{2}(\theta - i\,\pi)\,\exp[-i\,\gamma] \\ a &= \frac{1}{2}(\theta + i\,\pi)\,\exp[i\,\gamma] \end{aligned} \qquad \text{[III-65]}$$

the mode field (see Equation [II-24a]) b

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = i\,\hat{\mathbf{e}}\,\mathcal{E}\left\{ a\,\exp\left[i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - i\,\omega\,t\right] - a^\dagger\,\exp\left[-i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} + i\,\omega\,t\right]\right\} \qquad \text{[III-66a]}$$

becomes

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\,\hat{\mathbf{e}}\,\mathcal{E}\left\{ \pi\,\cos\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega\,t + \gamma\right) + \theta\,\sin\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega\,t + \gamma\right)\right\} \qquad \text{[III-66b]}$$

Since p has a coordinate space representation $-i\,\hbar\,d/d\,q = -i\left(\hbar\,\omega/2\right)^{1/2}\,d/d\,\theta$ and q has a momentum representation $i\,\hbar\,d/d\,p = i\left(\hbar/2\,\omega\right)^{1/2}\,d/d\,\pi$, [23]

$$\begin{aligned} \alpha\,a^\dagger - \alpha\,a &= \alpha_r\left(a^\dagger - a\right) + i\,\alpha_i\left(a^\dagger + a\right) \\ &= -\left[\alpha_r\,d/d\,\theta + \alpha_i\,d/d\,\pi\right] \end{aligned} \qquad \text{[III-67a]}$$

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22 Of course, in general $\left\langle (\Delta\mathbf{A})^2 \right\rangle \left\langle (\Delta\mathbf{B})^2 \right\rangle \geq \frac{1}{2}\left|\left\langle \left[\mathbf{A},\mathbf{B}\right]\right\rangle\right|^2$ where $\left\langle (\Delta\mathbf{A})^2 \right\rangle = \left\langle \mathbf{A}^2 \right\rangle - \left\langle \mathbf{A} \right\rangle^2$

23 If this unfamiliar, see Equations [I-20] and [I-22] in the lecture notes entitled *The Interaction of Radiation and Matter: Semiclassical Theory*.

and

$$\mathcal{A}^{\dagger}(\alpha) = \exp\left(\alpha\, a^{\dagger} - \alpha^{*}\, a\right) = \exp\left[-\left(\alpha_r\, d/d\theta + \alpha_i\, d/d\pi\right)\right] \qquad \text{[III-67b]}$$

Thus, $\mathcal{A}^{\dagger}(\alpha)$ defines or generates a two-dimensional Taylor expansion when it acts on a function of θ and π. In particular, if we take the “phase space” representation of the ground or vacuum state $|\theta\pi\rangle$ as the product of two Gaussians (see Equations [I-10a] and [I-29]), then $\mathcal{A}^{\dagger}(\alpha)|\theta\pi\rangle$ represents a shift or displacement of this “phase space” representation -- *i.e.*

$$\langle\theta\pi|\alpha\rangle = \langle\theta\pi|\mathcal{A}^{\dagger}(\alpha)|0\rangle = u_G(\theta - \alpha_r)\, u_G(\pi - \alpha_i) \qquad \text{[III-68]}$$

In light of Equation [II-23b], $|\alpha(t)\rangle = |\alpha \exp(-i\,\omega\, t)\rangle$ we can write

$$\langle\theta\pi|\alpha(t)\rangle = u_G\left(\theta - |\alpha|\cos(\omega\, t + \varphi)\right)\, u_G\left(\pi - |\alpha|\cos(\omega\, t + \varphi)\right) \qquad \text{[III-69]}$$

where $\alpha = |\,\alpha\,| \exp(i\,\varphi)$.

h. **The diagonal coherent-state representation of the density operator (Glauber-Sudarshan P-representation):**

It may be easily established that

$$\bar{\bar{\mathbf{1}}} = \frac{1}{\pi}\iint |\beta\rangle\langle\beta|\, d^2\beta = \iint |\beta\rangle\langle\beta|\, d\,\mathbf{Re}(\beta)\, d\,\mathbf{Im}(\beta) \qquad \text{[III-70]}$$

so that it seems quite reseasonable to look for a representation of the density matrix is the form

$$\bar{\bar{\rho}} = \iint P(\beta)\, |\beta\rangle\langle\beta|\, d^2\beta \qquad \text{[III-71]}$$

For a pure coherent state, P is clearly a two-dimensional delta function

Example 1 -- Coherent state

$$P(\beta) = \delta^{(2)}(\beta - \alpha) = \delta^{(1)}\big(\mathbf{Re}(\beta) - \mathbf{Re}(\alpha)\big)\,\delta^{(1)}\big(\mathbf{Im}(\beta) - \mathbf{Im}(\alpha)\big) \quad \text{[III-72]}$$

In general, using Equation [III-60] -- *i.e.*

$$\langle\beta|\alpha\rangle = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha\,\beta\right] \quad \text{[III-60]}$$

we may find a simple procedure for finding the P-representation by writing

$$\begin{aligned}\langle -\alpha|\,\overline{\rho}\,|\alpha\rangle &= \iint P(\beta)\,\langle -\alpha|\beta\rangle\langle\beta|\alpha\rangle\, d^2\beta \\ &= \exp\left(-|\alpha|^2\right)\iint\left[P(\beta)\,\exp\left(-|\beta|^2\right)\right]\exp\left[\alpha\beta - \beta\,\alpha\right] d^2\beta\end{aligned} \quad \text{[III-73]}$$

Thus, $\langle -\alpha|\,\overline{\rho}\,|\alpha\rangle\,\exp\left(-|\alpha|^2\right)$ is the two-dimensional Fourier transform 0f the function $P(\beta)\,\exp\left(-|\beta|^2\right)$ and we may write

$$P(\beta) = \frac{1}{\pi^2}\exp\left(|\beta|^2\right)\iint\langle -\alpha|\,\overline{\rho}\,|\alpha\rangle\,\exp\left(|\alpha|^2\right)\exp\left[-\alpha\beta + \beta\,\alpha\right] d^2\alpha \quad \text{[III-74]}$$

As a second example, consider a thermal radiation field described by a canonical ensemble

$$\overline{\rho} = \frac{\exp\left(-\mathcal{H}/k_B T\right)}{\mathrm{Tr}\left[\exp\left(-\mathcal{H}/k_B T\right)\right]} \quad \text{[III-75]}$$

where $\mathcal{H} = \hbar\omega\left\{a^\dagger a + \frac{1}{2}\right\}$. Thus,

$$\overline{\rho} = \sum_n\left[1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]\exp\left(-\frac{n\,\hbar\omega}{k_B T}\right)|n\rangle\langle n| \quad \text{[III-76]}$$

and
$$\langle n\rangle = \mathrm{Tr}\left[\bar{\rho}\; a^{\dagger} a\right] = \sum_n \left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^{-1} \qquad \text{[III-77]}$$

so that
$$\bar{\rho} = \sum_n \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}} \left|n\right\rangle\left\langle n\right| \qquad \text{[III-78]}$$

Thus, we can write
$$\left\langle n\right|\bar{\rho}\left|n\right\rangle = \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}} \qquad \text{[III-79]}$$

$$\left\langle -\alpha\right|\bar{\rho}\left|\alpha\right\rangle = \sum_n \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}} \left\langle -\alpha|n\right\rangle\left\langle n|\alpha\right\rangle$$

and
$$= \frac{\exp\left(-|\alpha|^2\right)}{1+\langle n\rangle} \sum_n \frac{\left(-|\alpha|^2\right)^n}{n!} \left[\frac{\langle n\rangle}{\left(1+\langle n\rangle\right)}\right]^n \qquad \text{[III-80]}$$

$$= \frac{\exp\left(-|\alpha|^2\right)}{1+\langle n\rangle} \exp\left[-|\alpha|^2 \Big/ \left(1+\frac{1}{\langle n\rangle}\right)\right]$$

Finally, we see that

Example 2 -- Thermal radiation - a chaotic state

$$P(\alpha) = \frac{\exp\left(|\alpha|^2\right)}{\pi^2\left(1+\langle n\rangle\right)} \iint \exp\left[-|\beta|^2 \Big/ \left(1+\frac{1}{\langle n\rangle}\right)\right] \exp\left(-\beta\alpha^* + \alpha\beta^*\right) d^2\beta$$
$$= \frac{1}{\pi\langle n\rangle} \exp\left(-|\alpha|^2/\langle n\rangle\right) \qquad \text{[III-81]}$$

As a third example, consider Fock or number state. From Equation [III-55] we see that

$$\langle -\alpha | \overline{\rho} | \alpha \rangle = \langle -\alpha | n \rangle \langle n | \alpha \rangle = \frac{\exp\left(-|\alpha|^2\right)}{n!} \left(-|\alpha|^2\right)^n \qquad \text{[III-82a]}$$

and

$$\begin{aligned} P(\beta) &= \frac{1}{n!} \frac{1}{\pi^2} \exp\left(|\beta|^2\right) \iint \left(-|\alpha|^2\right)^n \exp\left[-\alpha\beta + \beta\alpha\right] d^2\alpha \\ &= \frac{\exp\left(|\beta|^2\right)}{n!} \frac{\partial^{2n}}{\partial\beta^{*n}\partial\beta^{n}} \frac{1}{\pi^2} \iint \exp\left[-\alpha\beta + \beta\alpha\right] d^2\alpha \end{aligned} \qquad \text{[III-82b]}$$

so that

Example 3 -- Pure Fock or number state

$$P(\beta) = \frac{\exp\left(|\beta|^2\right)}{n!} \frac{\partial^{2n}}{\partial\beta^{*n}\partial\beta^{n}} \delta^{(2)}(\beta) \qquad \text{[III-82b]}$$

i. **The Glauber-Sudarshan-Klauder "optical equivalence" theorem:**

Suppose we have some "normally ordered" function

$$f^{(N)}\left(a, a^{\dagger}\right) = \sum_n \sum_m c_{nm}\, a^{\dagger n} a^{m} \qquad \text{[III-83]}$$

The expectation value is given by

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right) \right\rangle = \mathrm{Tr}\left[\overline{\rho}\, f^{(N)}\left(a, a^{\dagger}\right)\right] \qquad \text{[III-84]}$$

Using Equation [III-71] we see that

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right)\right\rangle = \mathrm{Tr}\left[\iint P(\alpha) \sum_n \sum_m c_{nm} |\alpha\rangle\langle\alpha| \, a^{\dagger n} \, a^{m} \, d^2\alpha\right]$$

$$= \iint P(\alpha) \sum_n \sum_m c_{nm} \langle\alpha| \, a^{\dagger n} \, a^{m} \, |\alpha\rangle \, d^2\alpha \qquad \text{[III-85a]}$$

$$= \iint P(\alpha) \sum_n \sum_m c_{nm} \, \alpha^{*n} \, \alpha^{m} \, d^2\alpha$$

or, finally, the **“optical equivalence” theorem**

$$\left\langle f^{(N)}\left(a, a^{\dagger}\right)\right\rangle = \iint P(\alpha) \, f^{(N)}\left(\alpha, \alpha^{*}\right) \qquad \text{[III-85b]}$$

j. **The Uncertainty Relationship for $\{\theta, \pi\}$:**

Since $\left[a, a^{\dagger}\right] = 1$ we see from Equation [III-64a] that

$$\left\langle \Delta\theta^2 \right\rangle = \left\langle \theta^2 \right\rangle - \left\langle \theta \right\rangle^2$$

$$= \left\langle a^{\dagger} \, a^{\dagger} \right\rangle \exp\left[2\,i\,\gamma\right] + \left\langle a \; a \right\rangle \exp\left[-2\,i\,\gamma\right] + \left\langle a^{\dagger} \; a \right\rangle + \left\langle a \; a^{\dagger} \right\rangle$$
$$- \left\langle a^{\dagger} \right\rangle^2 \exp\left[2\,i\,\gamma\right] - \left\langle a \right\rangle^2 \exp\left[2\,i\,\gamma\right] - 2\left\langle a^{\dagger} \right\rangle \left\langle a \right\rangle \qquad \text{[III-86]}$$

$$= \left\langle : \Delta\,\theta^2 : \right\rangle + 1$$

where $\langle : \mathbf{A} : \rangle$ symbollizes the normally ordered expectation value of the operator $\mathbf{A}$. From Equation [III-85b]

$$\left\langle : \Delta\theta^2 \left(a, a^{\dagger}\right) : \right\rangle = \iint P(\alpha) \, \Delta\theta^2\left(\alpha, \alpha^{*}\right) d^2\alpha \qquad \text{[III-87]}$$

$$\left\langle :\Delta\theta^2\left(a, a^\dagger\right):\right\rangle = \iint P(\alpha)\left[\Delta\alpha^* \exp(i\gamma) + \Delta\alpha \exp(-i\gamma)\right]^2 d^2\alpha \quad \text{[III-88]}$$

If we choose γ (and $P(\alpha)$) such that $\left\langle :\Delta\theta^2\left(a, a^\dagger\right):\right\rangle < 0$, then $\langle\Delta\theta^2\rangle > 1$ and $\langle\Delta\pi^2\rangle > 1$ (**squeezed states**)!