III. REPRESENTATIONS OF PHOTON STATES

1. Fock or "Number" States: 11

As we have seen, the Fock or number states

\[ |\{n_\mathbf{k}\}\rangle = \prod_{\sigma} |n_{\mathbf{k}, \sigma}\rangle \]  \[ \text{III-1} \]

are complete set eigenstates of an important group of commuting observables -- viz. \( \mathcal{H}_{rad} \), \( \mathcal{N} \), and \( \mathcal{M} \).

Reprise of Characteristics and Properties of Fock States:

a. The expectation value of the number operator and the fractional uncertainty associated with a single Fock state:

\[ \langle n \rangle_\mathcal{N} |n\rangle = n \]  \[ \text{III-2a} \]

\[ \Delta n = \text{"uncertainty"} = \sqrt{\langle n \rangle_\mathcal{N}^2 |n\rangle - \langle n | \mathcal{N} |n\rangle^2} = 0 \]  \[ \text{III-2b} \]

b. Expectation value of the fields associated with a single mode:

For one mode Equations [II-24a] and [II-24b] reduce to

\[ \tilde{E} (\mathbf{r}, t) = i \tilde{\mathbf{e}} \mathcal{E} \{ a \exp [i \mathbf{k} \cdot \mathbf{r} - i \omega t] - a^\dagger \exp [-i \mathbf{k} \cdot \mathbf{r} - i \omega t] \} \]  \[ \text{III-3a} \]

\[ \tilde{H} (\mathbf{r}, t) = i \sqrt{\frac{\mathcal{E}}{\mu_0}} \mathcal{E} \{ \mathbf{k} \times \tilde{\mathbf{e}} \} \{ a (t) \exp [i \mathbf{k} \cdot \mathbf{r} - i \omega t] - a^\dagger (t) \exp [-i \mathbf{k} \cdot \mathbf{r} - i \omega t] \} \]  \[ \text{III-3b} \]

11 In what follows, for simplicity we drop the \( \hat{\mathbf{k}} \) subscripts on the operators and state vectors with the obvious meaning that \(|\{n_\mathbf{k}\}\rangle \Rightarrow |n\rangle\), \( a_\mathbf{k} \Rightarrow a \), etc...
where \( \mathcal{E} = \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \)

\[
\langle n | \vec{E} | n \rangle = 0 \\
\langle n | \vec{H} | n \rangle = 0
\]  

\[
\Delta E = \sqrt{\{\langle n | \vec{E} \cdot \vec{E} | n \rangle - \langle n | \vec{E} | n \rangle^2\}} = \frac{\hbar \omega}{\varepsilon_0 V} (n + \frac{1}{2})^\frac{1}{2} = \sqrt{2} \mathcal{E} (n + \frac{1}{2})^\frac{1}{2}
\]

\[
\Delta H = \sqrt{\{\langle n | \vec{H} \cdot \vec{H} | n \rangle - \langle n | \vec{H} | n \rangle^2\}} = \frac{\hbar \omega}{\mu_0 V} (n + \frac{1}{2})^\frac{1}{2} = \frac{\varepsilon_0}{\mu_0} \sqrt{2} \mathcal{E} (n + \frac{1}{2})^\frac{1}{2}
\]  

\[
\Delta E \Delta H = c \frac{\hbar \omega}{V} (n + \frac{1}{2}) = \sqrt{\frac{\varepsilon_0}{\mu_0}} 2 \mathcal{E}^2 (n + \frac{1}{2})
\]

c. Phase of field associated with single mode:

To obtain something analogous to the classical theory we would like to separate the creation and destruction operators (and, thus, the electric and magnetic field operators) into a product of amplitude and phase operators. Following Susskind and Glogower,\(^{12}\) we define a phase operator, \( \Phi \) such that

\[
a \equiv (\mathcal{N} + 1)^\frac{1}{2} \exp(i \Phi) \\
\hat{a} \equiv \exp(-i \Phi) (\mathcal{N} + 1)^\frac{1}{2}
\]  

[ III-5 ]

Defined in this way, the basic properties of the phase operator may be evaluated from known properties of the creation, destruction and number operators.

Inverting, we obtain

\(^{12}\) Susskind, L. and Glogower, J., *Physics*, 1, 49 (1964)
\[ \exp(i \Phi) \equiv (\mathcal{N} + 1)^{\frac{i}{\hbar}} a \]
\[ \exp(-i \Phi) \equiv a^\dagger (\mathcal{N} + 1)^{\frac{-i}{\hbar}} \]  

[III-6]

and since \( a a^\dagger = \mathcal{N} + 1 \), it follows that
\[ \exp(i \Phi) \exp(-i \Phi) = 1 \]  

[III-7]

**but only in this order!** Operating on number states with the phase operators, we obtain from Equation [I-26]
\[ \exp(i \Phi) |n\rangle = (\mathcal{N} + 1)^{\frac{i}{\hbar}} a |n\rangle = (\mathcal{N} + 1)^{\frac{i}{\hbar}} (n)^n |n-1\rangle = |n-1\rangle \]
\[ \exp(-i \Phi) |n\rangle = a^\dagger (\mathcal{N} + 1)^{\frac{-i}{\hbar}} |n\rangle = a^\dagger (n+1)^n |n\rangle = |n+1\rangle \]  

[III-8]

Consequently, the **only nonvanishing matrix elements** of the phase operator are
\[ \langle n-1 | \exp(i \Phi) | n \rangle = 1 \]
\[ \langle n+1 | \exp(-i \Phi) | n \rangle = 1 \]  

[III-9]

The phase operators defined by Equation [III-36] do have the felicitous or **classically analogous** property of revealing **magnitude independent** information, but unfortunately they are nonHermitian operators -- *i.e.*
\[ \langle n-1 | \exp(i \Phi) | n \rangle \neq \langle n | \exp(i \Phi) | n-1 \rangle^* \]

-- and, hence, **cannot represent observables**. However, they may be **paired** into operators that are observables -- *viz.*
\[
\cos \Phi = \frac{1}{2} \left\{ \exp(i \Phi) + \exp(-i \Phi) \right\} \\
\sin \Phi = \frac{1}{2i} \left\{ \exp(i \Phi) - \exp(-i \Phi) \right\}
\]

which have the following nonvanishing matrix elements:

\[
\langle n - 1 | \cos \Phi | n \rangle = \langle n | \cos \Phi | n - 1 \rangle = \frac{1}{2} \\
\langle n - 1 | \sin \Phi | n \rangle = -\langle n | \sin \Phi | n - 1 \rangle = \frac{1}{2i}
\]

These *nearly commuting* operators may be adopted as the quantum mechanical operators which represent (as we will demonstrate anon) the observable phase properties of the electromagnetic field.

For the Fock state:

\[
\langle n | \cos \Phi | n \rangle = \langle n | \sin \Phi | n \rangle = 0
\]

\[
\Delta \cos \Phi = \Delta \sin \Phi = \sqrt{\left\{ \langle n | \cos^2 \Phi | n \rangle - \langle n | \cos \Phi | n \rangle^2 \right\}} = \sqrt{\frac{1}{2}}
\]

\[
\Delta \cos \Phi \Delta \sin \Phi = \frac{1}{2}
\]

c. The coordinate or Schrödinger representation of state:

Recall from Equations [I-10a] and [I-31] that

\[
\langle n | [\cos \Phi, \sin \Phi] | n' \rangle = \frac{i}{2} \delta_{nn'} \delta_{n0}
\]

---

13 Also, it may be easily established that the matrix elements of their commutator are given by
\[
\langle q|n\rangle = \frac{1}{\sqrt{n!}} \left[ \sqrt{\frac{m}{2\hbar\omega}} \right]^n \left[ \omega q - \frac{\hbar d}{m dq} \right]^n \langle q|0\rangle \\
= \sqrt{\frac{1}{2^n n!}} \frac{\omega}{\pi \hbar} H_n \left( \sqrt{\frac{m\omega}{\hbar}} q \right) \exp \left[ -\frac{m\omega}{2\hbar} q^2 \right]
\]

Therefore, the probability \( P(q) \) of eigenvalues \( q \) for a given Fock state \( |n\rangle \) is given by

\[
P(q) = \langle q|n\rangle \langle n|q\rangle = \frac{1}{2^n n!} \frac{\omega}{\pi \hbar} H_n^2 \left( \sqrt{\frac{m\omega}{\hbar}} q \right) \exp \left[ -\frac{m\omega}{2\hbar} q^2 \right]
\]

**d. Approximate “localization” of a photon:**

Of course a plane wave is distributed or “de-localized” in both time and space. Defining the “wave function for a photon” is a task fraught with danger, but the simpler task of defining a wave function approximately localized at a given instant is relatively straightforward -- viz.

\[
\left| \psi \left( \mathbf{r}_0 \right) \right|_{\mathcal{K}_0} = C \sum_k \exp \left[ -\frac{|\mathbf{k} - \mathbf{k}_0|^2}{2 \Delta \mathbf{k}} \right] \left| \exp \left[ i \mathbf{k}_0 \cdot \mathbf{r}_0 \right] |0,0,0,\ldots n_k = 1,\ldots,0,0,0\rangle\right]
\]

**2. Photon States of Well-defined Phase:**

Consider the state defined by

\[
|\psi\rangle = \lim_{s \to \infty} (s + 1)^{-\zeta} \sum_{n=0}^{s} \exp \left[ in \varphi \right] |n\rangle
\]

---


Clearly, \( \langle \phi | \phi \rangle = 1 \) given the orthonormal properties of the number states. Essential question: Is this state an eigenstate of the phase operators? To answer the question we need to consider the following **potential eigenvalue equation**:

\[
\cos \Phi \, | \phi \rangle = \frac{1}{2} \lim_{s \to \infty} (s + 1)^{-\frac{\nu}{2}} \left\{ \sum_{n=0}^{s} \exp[i \, n \, \varphi] \exp[i \, \Phi] | n \rangle + \sum_{n=0}^{s} \exp[i \, n \, \varphi] \exp[-i \, \Phi] | n \rangle \right\} \quad [\text{III-16a}]
\]

Using Equations [III-10] and [III-10], we obtain

\[
\cos \Phi \, | \phi \rangle = \frac{1}{2} \lim_{s \to \infty} (s + 1)^{-\frac{\nu}{2}} \left\{ \sum_{n=1}^{s} \exp[i \, n \, \varphi] | n - 1 \rangle + \sum_{n=0}^{s} \exp[i \, n \, \varphi] | n + 1 \rangle \right\}
\]

\[
= \frac{1}{2} \lim_{s \to \infty} (s + 1)^{-\frac{\nu}{2}} \left\{ \exp(i \varphi) \sum_{\nu=0}^{s-1} \exp[i \, \nu \, \varphi] | \nu \rangle + \exp[-i \varphi] \sum_{\nu=1}^{s+1} \exp[i \, \nu \, \varphi] | \nu \rangle \right\} \quad [\text{III-16b}]
\]

\[
= \cos \varphi \, | \phi \rangle + \frac{1}{2} \lim_{s \to \infty} (s + 1)^{-\frac{\nu}{2}} \left\{ \exp[i \, s \, \varphi] | s + 1 \rangle - \exp[i \, (s + 1) \, \varphi] | s \rangle - \exp[-i \, \varphi] | 0 \rangle \right\}
\]

so that the state \( | \phi \rangle \) fails to be a strict eigenket of \( \cos \Phi \) by terms that diminish faster than \( (s + 1)^{-\frac{\nu}{2}} \) as \( s \to \infty \). Similarly, we can see that diagonal matrix elements of \( \cos \Phi \) and \( \sin \Phi \) are given by

\[
\langle \phi | \cos \Phi | \phi \rangle = \cos \varphi \left\{ 1 - \lim_{s \to \infty} (s + 1)^{-1} \right\} \quad \Rightarrow \quad \cos \varphi \quad [\text{III-17a}]
\]

\[
\langle \phi | \sin \Phi | \phi \rangle = \sin \varphi \left\{ 1 - \lim_{s \to \infty} (s + 1)^{-1} \right\} \quad \Rightarrow \quad \sin \varphi \quad [\text{III-17b}]
\]
Reprise of Characteristics and Properties of Phase States:

a. The expectation value of the number operator and the fractional uncertainty associated with a state of well-defined phase:

\[
\langle \phi | \hat{N} | \phi \rangle = \lim_{s \to \infty} (s + 1)^{-1} \sum_{n=0}^{s} n = \lim_{s \to \infty} (s + 1)^{-1} \left[ \frac{s(s+1)}{2} \right] = \lim_{s \to \infty} \frac{s}{2}
\]

\[
\left[ \text{fractional uncertainty} \right] = \sqrt{\frac{\langle \phi | \hat{N}^2 | \phi \rangle - \langle \phi | \hat{N} | \phi \rangle^2}{\langle \phi | \hat{N} | \phi \rangle}}
\]

\[
= \sqrt{\lim_{s \to \infty} (s + 1)^{-1} \sum_{n=0}^{s} n^2 - \left( \lim_{s \to \infty} (s + 1)^{-1} \sum_{n=0}^{s} n \right)^2}
\]

\[
= \lim_{s \to \infty} (s + 1)^{-1} \sum_{n=0}^{s} n
\]

\[
= \sqrt{\lim_{s \to \infty} \left[ \frac{1}{6} (2s^2 + s) - \frac{1}{4} s^2 \right]}
\]

\[
= \lim_{s \to \infty} \frac{s}{2} = \frac{1}{\sqrt{3}}
\]  

[ III-18a ]

[ III-18b ]

b. Expectation value of the fields associated with a single mode:

From Equation [ III-3a ]

\[
\langle \phi | \vec{E} | \phi \rangle = -2 \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \hat{e} \sin(\vec{k} \cdot \vec{r} - \omega t + \phi) \lim_{s \to \infty} (s + 1)^{-1} \sum_{n=0}^{s} (n + 1)^2
\]

\[
\Rightarrow \text{diverges as } \sqrt{s} \text{ for large } s!
\]

[ III-19 ]
c. Phase of field associated with single mode:

\[ \langle \varphi | \cos \Phi | \varphi \rangle = \cos \varphi \]
\[ \langle \varphi | \sin \Phi | \varphi \rangle = \sin \varphi \]  

\[ \Delta \cos \Phi = \Delta \sin \Phi = \sqrt{\langle \varphi | \cos^2 \Phi | \varphi \rangle - \langle \varphi | \cos \Phi | \varphi \rangle^2} = 0 \]  

\[ \text{[ III-20a]} \]

\[ \text{[ III-20b]} \]

d. Probability of photon number:

Finally, we may easily deduce the probability of finding \( n \) photons (i.e. the photon statistics) in a particular state of well defined phase -- \( \text{viz.} \)

\[ P_n = \langle n | \varphi \rangle^2 \equiv \lim_{s \to \infty} (s + 1)^{-1} \]  

\[ \text{[ III-50]} \]

We see that there is a equal, but small probability of any number: this agrees with the intuition that the magnitude of the field is completely undetermined if the phase is precisely known!

3. Coherent Photon States:  

It would, indeed, be useful to have eigenstates of the destruction operator (electric or magnetic field) -- \( \text{viz.} \)

\[ a_k \left| \alpha_k \right\rangle = \alpha_k \left| \alpha_k \right\rangle \]  

\[ \text{[ III-51]} \]

**Reprise of Characteristics and Properties of Coherent States:**

a. The Fock state representation of the coherent state:

\[ \text{[ III-52]} \]

\[ \text{[ III-53]} \]
Since \( a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \) and \( a a^\dagger = n + 1 \), then \( \langle n| a = \sqrt{n+1} \langle n+1| \) and we are able to write a **representative** of the sought state in the number state basis -- *viz.*

\[
\langle n| a |\alpha\rangle = \sqrt{n+1} \langle n+1|\alpha\rangle = \alpha \langle n| \alpha\rangle
\]

[III-52a]

or

\[
\langle n| \alpha\rangle = \frac{\alpha}{\sqrt{n}} \langle n-1|\alpha\rangle = \frac{n!}{\sqrt{n}!} \langle 0|\alpha\rangle
\]

[III-52b]

Using the expansion of the identity operator, the eigenket becomes

\[
|\alpha\rangle = \sum_n \langle n| \alpha\rangle |n\rangle = \langle 0|\alpha\rangle \sum_n \frac{n!^n}{n!} |n\rangle.
\]

[III-53]

To normalize the eigenket write

\[
\langle \alpha|\alpha\rangle = \langle \alpha| 0 \rangle \langle 0|\alpha\rangle \sum_n \frac{n!^n}{n!} = \langle \alpha| 0 \rangle \langle 0|\alpha\rangle \exp\left[|\alpha|^2\right] = 1
\]

[III-54]

so that \( \langle \alpha| 0 \rangle = \langle 0|\alpha\rangle = \exp\left[\left(-\frac{1}{2} |\alpha|^2\right)\right]. \) Finally, we see that

\[
|\alpha\rangle = \exp\left[-\frac{1}{2} |\alpha|^2\right] \sum_n \frac{n!^n}{\sqrt{n!}} |n\rangle
\]

[III-55]

is a normalize representation of the eigenkets of the destruction operator.
b. The expectation value of the number operator and the fractional uncertainty associated with a coherent state:

\[ \langle \alpha | \mathcal{N} | \alpha \rangle = |\alpha|^2 \]  

[ III-56a ]

\[
\text{fractional uncertainty} = \frac{\sqrt{\{\langle \alpha | \mathcal{N}^2 | \alpha \rangle - \langle \alpha | \mathcal{N} | \alpha \rangle^2\}}}{\langle \alpha | \mathcal{N} | \alpha \rangle} = \frac{1}{|\alpha|^2} \sqrt{\exp(-|\alpha|^2) \sum \frac{|\alpha|^2n}{n!} [n(n-1)+n] - |\alpha|^4} 
\]

[ III-56b ]

\[
= \frac{1}{|\alpha|^2} \sqrt{\exp(-|\alpha|^2) \sum \frac{|\alpha|^2n}{n!} [n(n-1)+n] - |\alpha|^4} 
\]

\[
= |\alpha|^{-1} 
\]

Thus, we see that the fractional uncertainty diminishes with mean photon number!

c. Expectation value of the electric field associated with a single mode:

From Equation [ III-3a ]

\[
\langle \alpha | \hat{E} | \alpha \rangle = -2 \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \hat{e} |\alpha| \sin(\vec{k} \cdot \vec{r} - \omega t + \vartheta) 
\]

[ III-57a ]

where \( \alpha = |\alpha| \exp(i \vartheta) \).

\[
\Delta E = \sqrt{\{ \langle \alpha | \vec{E} \cdot \vec{E} | \alpha \rangle - \langle \alpha | \vec{E} | \alpha \rangle^2 \}} = \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \]

[ III-57b ]

\[ \Delta H = \frac{1}{c \mu_0} \sqrt{\frac{\hbar \omega}{2 \varepsilon_0 V}} \] for the coherent state, so that \( \Delta E \Delta H = c \hbar \omega / 2 V \).

R. Victor Jones, May 2, 2000
d. **Probability of photon number:**

From the representation of the coherent state given in Equation [III-55] we may easily deduce the probability of finding \( n \) photons (the photon statistics) in a particular coherent state is given by a **Poisson distribution** characterized by the mean value \( \bar{n} = |\alpha|^2 \). -- *viz.*

\[
P_n = \langle n | \alpha \rangle^2 = \exp\left[-|\alpha|^2\right] \frac{|\alpha|^2^n}{n!}
\]  

[III-58]

**Sample Poisson Distributions - Coherent State Photon Statistics**
e. Phase of field associated with single mode:

\[
\langle \alpha | \cos \Phi | \alpha \rangle = \frac{1}{2} \exp \left[ -\frac{1}{2} |\alpha|^2 \right] \sum_{n} \sum_{n'} \{n' | \frac{\alpha^{n'} + \alpha^{*n'}}{\sqrt{n!}} \} \frac{\alpha^{n}}{\sqrt{(n+1)!}} |n\rangle \\
= \frac{1}{2} \exp \left[ -\frac{1}{2} |\alpha|^2 \right] \sum_{n} \left[ \frac{\alpha^{n} + \alpha^{*n}}{\sqrt{(n+1)!}} \right] \left[ \frac{\alpha^{n}}{\sqrt{n!}} \right] \\
= |\alpha| \cos \theta \exp \left[ -\frac{1}{2} |\alpha|^2 \right] \sum_{n} \left[ \frac{|\alpha|^2}{n!} \right] \left[ \frac{1}{\sqrt{(n+1)!}} \right] \\
\]

[ III-59a]

Unfortunately, it is not possible to evaluate this summation analytically. However, Carruthers\textsuperscript{18} has given an asymptotic expansion which is valid for a large mean number of photons -- \textit{viz.}

\[
\langle \alpha | \cos \Phi | \alpha \rangle = \cos \theta \left[ 1 - \frac{1}{8 |\alpha|^2} + \ldots \right] \quad |\alpha|^2 >> 1 \\
[ III-59b]
\]

f. Coherent states as a basis:

As we will see presently, the coherent states are very useful in describing the quantized electromagnetic field, but, alas, there is a complication -- the coherent states are not truly orthogonal! From Equation [ III-6 ] we see that

\[
\langle \beta | \alpha \rangle = \exp \left[ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 \right] \sum_{n} \frac{\beta^{*n} \alpha^{n}}{n!} \\
= \exp \left[ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \alpha \beta^{*} \right] \\
[ III-60 ]
\]

so that

\[ \langle \alpha \beta \rangle \langle \beta | \alpha \rangle = \exp \left( -|\alpha|^2 - |\beta|^2 + \alpha \beta^* + \alpha^* \beta \right) \\
= \exp \left( - (\alpha - \beta)(\alpha^* - \beta^*) \right) = \exp \left( - \left| \alpha - \beta \right|^2 \right) \]

That is, the eigenkets are approximately orthogonal only when \(|\alpha - \beta|\) is large!

g. The “displacement operator:”

There are a growing and significant set of applications where it is useful to express the coherent states directly in terms of the vacuum state \(|0\rangle\). If we use the number state generating rule

\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \]

-- i.e. Equation [ I-27 ] -- the coherent state may be written in the form

\[ |\alpha\rangle = \exp \left( - \frac{1}{2} |\alpha|^2 \right) \sum_n \frac{(\alpha a^\dagger)^n}{n!} |0\rangle = \exp \left( \alpha a^\dagger - \frac{1}{2} |\alpha|^2 \right) |0\rangle \]

[ III-62 ]

If we make use of the Baker-Hausdorff theorem,\(^{19}\) we may easily show that

\(^{19}\) The Baker-Hausdorff theorem or identity may be stated as

\[ \exp \{ A + B \} = \exp \{ A \} \exp \{ B \} \exp \{ - \frac{1}{2} [ A , B ] \} \]

when \([ A , [ A , B ] ] = [ B , [ A , B ] ] = 0\). For a proof, see, for example, Charles P. Slichter’s *Principles of Magnetic Resonance*, Appendix A or William Louisell’s *Radiation and Noise in Quantum Electronics.*

\[ \text{R. Victor Jones, May 2, 2000} \]
\[ |\alpha\rangle = \mathcal{A}^\dagger (\alpha) |0\rangle = \exp(\alpha a^\dagger - \alpha^* a) |0\rangle \]  

[ III-63 ]

so that \( \mathcal{A}^\dagger (\alpha) \) may be interpreted as a creation operator which generates a coherent state from the vacuum. (Its adjoint operator \( \mathcal{A} (\alpha) = \mathcal{A}^\dagger (-\alpha) \) is a destruction operator which destroys a state). In some treatments \( \mathcal{A}^\dagger (\alpha) \) is described as the “displacement operator” (written \( \mathcal{D} (\alpha) \))\(^{20}\) and the coherent states are called the “displaced states of the vacuum.”\(^ {21}\)

To explore this point of view (and to give some meaning to the phase of the coherent state eigenvalue), we may express \( |\alpha\rangle \) in a two-dimensional, dimensionless “phase space” representation. To that end, following Equation [ I-16 ], we write the dimensionless coordinate as

\[ \theta = \left( \frac{2 m \omega}{\hbar} \right)^{\frac{1}{2}} \gamma = a^\dagger \exp[i \gamma] + a \exp[-i \gamma] \]  

[ III-64a ]

and the dimensionless momentum as

\[ \pi = \left( \frac{2}{m \hbar \omega} \right)^{\frac{1}{2}} \gamma = a^\dagger \exp[i(\gamma + \pi/2)] + a \exp[-i(\gamma + \pi/2)] \]  

[ III-64b ]

so that

\[ [\theta, \pi] = 2i [a, a^\dagger] = 2i \]  

[ III-64c ]

\(^{20}\) We can (or rather you will) show that \( \mathcal{D}^\dagger (\alpha) a \mathcal{D} (\alpha) = a + \alpha \) and \( \mathcal{D}^\dagger (\alpha) a^\dagger \mathcal{D} (\alpha) = a^\dagger + \alpha^* \)

and since these variables are canonical \(^{22}\)

\[
\left\langle (\Delta \theta)^2 \right\rangle \left\langle (\Delta \pi)^2 \right\rangle \geq 1 \tag{III-64d}
\]

Since

\[
a^\dagger = \frac{1}{2} (\theta - i \pi) \exp[-i \gamma] \tag{III-65}
\]

\[
a = \frac{1}{2} (\theta + i \pi) \exp[i \gamma]
\]

the mode field (see Equation [II-24a]) b

\[
\tilde{E}(\mathbf{r}, t) = i \hat{e} \mathcal{E} \left\{ a \exp \left[ i \mathbf{k} \cdot \mathbf{r} - i \omega t \right] - a^\dagger \exp \left[ -i \mathbf{k} \cdot \mathbf{r} + i \omega t \right] \right\} \tag{III-66a}
\]

becomes

\[
\tilde{E}(\mathbf{r}, t) = - \hat{e} \mathcal{E} \left\{ \pi \cos \left( \mathbf{k} \cdot \mathbf{r} - \omega t + \gamma \right) + \theta \sin \left( \mathbf{k} \cdot \mathbf{r} - \omega t + \gamma \right) \right\} \tag{III-66b}
\]

Since \( p \) has a coordinate space representation \(-i \hbar d/dq = -i (\hbar \omega/2) \hat{k} \cdot d/d\theta \)

and \( q \) has a momentum representation \( i \hbar d/dp = i (\hbar/2 \omega) \hat{k} \cdot d/d\pi \), \(^{23}\)

\[
\alpha a^\dagger - \alpha^* a = \alpha \left[ a^\dagger - a \right] + i \alpha \left[ a^\dagger + a \right]
\]

\[
= -[\alpha, d/d\theta + \alpha, d/d\pi] \tag{III-67a}
\]

---

\(^{22}\) Of course, in general \( \left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{2} \left\langle [A, B] \right\rangle^2 \) where \( \left\langle (\Delta A)^2 \right\rangle = \langle A^2 \rangle - \langle A \rangle^2 \)

\(^{23}\) If this unfamiliar, see Equations [I-20] and [I-22] in the lecture notes entitled *The Interaction of Radiation and Matter: Semiclassical Theory.*

R. Victor Jones, May 2, 2000
and

\[ \mathcal{A}^\dagger(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right) = \exp\left[-(\alpha_c \, \frac{d}{d\theta} + \alpha_i \, \frac{d}{d\pi}\right] \]  

[ III-67b ]

Thus, \( \mathcal{A}^\dagger(\alpha) \) defines or generates a two-dimensional Taylor expansion when it acts on a function of \( \theta \) and \( \pi \). In particular, if we take the “phase space” representation of the ground or vacuum state \( |\theta\pi\rangle \) as the product of two Gaussians (see Equations [ I-10a ] and [ I-29 ]), then \( \mathcal{A}^\dagger(\alpha) |\theta\pi\rangle \) represents a shift or displacement of this “phase space” representation -- i.e.

\[ \langle\theta\pi|\alpha\rangle = \langle\theta\pi|\mathcal{A}^\dagger(\alpha)|0\rangle = u_c(\theta - \alpha_c) \, u_c(\pi - \alpha_i) \]  

[ III-68 ]

In light of Equation [ II-23b ], \( |\alpha(t)\rangle = |\alpha \exp(-i \, \omega t)\rangle \) we can write

\[ \langle\theta\pi|\alpha(t)\rangle = u_c(\theta - |\alpha| \cos(\omega \, t + \phi)) \, u_c(\pi - |\alpha| \cos(\omega \, t + \phi)) \]  

[ III-69 ]

where \( \alpha = |\alpha| \exp(i \, \phi) \).

h. The diagonal coherent-state representation of the density operator
(Glauber-Sudarshan P-representation):

It may be easily established that

\[ \overline{1} = \frac{1}{\pi} \int \int |\beta\rangle \langle\beta| \, d^2 \beta = \int \int |\beta\rangle \langle\beta| \, d \text{Re}(\beta) \, d \text{Im}(\beta) \]  

[ III-70 ]

so that it seems quite reasonable to look for a representation of the density matrix in the form

\[ \overline{\rho} = \int \int \rho(\beta) |\beta\rangle \langle\beta| \, d^2 \beta \]  

[ III-71 ]

For a pure coherent state, \( \rho \) is clearly a two-dimensional delta function

R. Victor Jones, May 2, 2000
Example 1 -- Coherent state

\[ P(\beta) = \delta^{(2)}(\beta - \alpha) = \delta^{(1)}(\text{Re}(\beta) - \text{Re}(\alpha)) \delta^{(1)}(\text{Im}(\beta) - \text{Im}(\alpha)) \]  

[III-72]

In general, using Equation [III-60] -- i.e.

\[ \langle \beta|\alpha \rangle = \exp \left[ -\frac{1}{2} |\alpha|^2 - \frac{1}{2} \beta \right]^2 + \alpha \beta^* \]

[III-60]

we may find a simple procedure for finding the P-representation by writing

\[ \langle -\alpha|\beta \rangle = \int P(\beta) \langle -\alpha|\beta \rangle |\beta \rangle d^2 \beta \]

\[ = \exp(-|\alpha|^2) \int \left[ P(\beta) \exp(-|\beta|^2) \right] \exp[\alpha \beta^* - \beta \alpha^*] d^2 \beta \]

[III-73]

Thus, \( \langle -\alpha|\beta \rangle \exp(-|\alpha|^2) \) is the two-dimensional Fourier transform of the function \( P(\beta) \exp(-|\beta|^2) \) and we may write

\[ P(\beta) = \frac{1}{\pi} \exp(|\beta|^2) \int \langle -\alpha|\betab \rangle \langle \alpha \rangle \exp\left(|\alpha|^2\right) \exp[-\alpha \beta^* + \beta \alpha^*] d^2 \alpha \]

[III-74]

As a second example, consider a thermal radiation field described by a canonical ensemble

\[ \rho = \exp(-H/k_B T) \]

[III-75]

where \( H = \hbar \omega \left\{ a^\dagger a + \frac{1}{2} \right\} \). Thus,

\[ \bar{\rho} = \sum_n \left[ 1 - \exp\left( \frac{\hbar \omega}{k_B T} \right) \right] \exp\left( -\frac{n \hbar \omega}{k_B T} \right) |n \rangle \langle n | \]

[III-76]
and
\[ \langle n \rangle = \text{Tr} \left[ \overline{\rho} \ a^\dagger a \right] = \sum_n \left[ \exp \left( \frac{\hbar \omega}{k_B T} \right)^{-1} \right] \]

so that
\[ \overline{\rho} = \sum_n \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^n+1} |n\rangle \langle n| \]

Thus, we can write
\[ \langle n | \overline{\rho} | n \rangle = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}} \]

\[ \langle -\alpha | \overline{\rho} | \alpha \rangle = \sum_n \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^n+1} \langle -\alpha | n \rangle \langle n | \alpha \rangle \]

and
\[ = \exp \left( -\frac{|\alpha|^2}{1 + \langle n \rangle} \right) \sum \frac{(-1)^n}{n!} \left[ \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^n+1} \right] \]

Finally, we see that

**Example 2 -- Thermal radiation - a chaotic state**

\[ P(\alpha) = \frac{\exp \left( \frac{|\alpha|^2}{\pi^2 (1 + \langle n \rangle)} \right)}{\pi^2 (1 + \langle n \rangle)} \left[ \exp \left( -\frac{|\alpha|^2}{1 + \langle n \rangle} \right) \right] \exp \left( -\beta \alpha^* + \alpha \beta^* \right) d^2 \beta \]

\[ = \frac{1}{\pi \langle n \rangle} \exp \left( -\frac{|\alpha|^2}{\langle n \rangle} \right) \]

As a third example, consider Fock or number state. From Equation [ III-55 ] we see that
\begin{align*}
\langle -\alpha | \bar{\Phi} | \alpha \rangle = \frac{\exp\left(-|\alpha|^2\right)}{n!} (-|\alpha|^2)^n \tag{III-82a}
\end{align*}

and

\begin{align*}
P(\beta) &= \frac{1}{n!} \frac{1}{\pi^2} \exp\left(|\beta|^2\right) \iint (-|\alpha|^2)^n \exp\left[-\alpha\beta^* + \beta\alpha^*\right] d^2 \alpha \\
&= \frac{\exp\left(|\beta|^2\right)}{n!} \frac{\partial^{2n}}{\partial \beta^* \partial \beta^n} \frac{1}{\pi^2} \iint \exp\left[-\alpha\beta^* + \beta\alpha^*\right] d^2 \alpha \tag{III-82b}
\end{align*}

so that

**Example 3 -- Pure Fock or number state**

\begin{align*}
P(\beta) &= \frac{\exp\left(|\beta|^2\right)}{n!} \frac{\partial^{2n}}{\partial \beta^* \partial \beta^n} \delta^{(2)}(\beta) \tag{III-82b}
\end{align*}

i. **The Glauber-Sudarshan-Klauder “optical equivalence” theorem:**

Suppose we have some “normally ordered” function

\begin{align*}
f^{(N)}(a, a^\dagger) &= \sum_n \sum_m c_{nm} a^n a^m \tag{III-83}
\end{align*}

The expectation value is given by

\begin{align*}
\left\langle f^{(N)}(a, a^\dagger) \right\rangle = \text{Tr} \left[ \bar{\Phi} f^{(N)}(a, a^\dagger) \right] \tag{III-84}
\end{align*}
Using Equation \[ \text{III-71} \] we see that
\[
\left\langle f^{(N)}(a, a^\dagger) \right\rangle = \text{Tr} \left[ \int \int P(\alpha) \sum_n \sum_m c_{nm} \langle \alpha | a^m a^n | \alpha \rangle \, d^2 \alpha \right]
\]
\[
= \int \int P(\alpha) \sum_n \sum_m c_{nm} \langle \alpha | a^m a^n | \alpha \rangle \, d^2 \alpha
\]
\[
= \int \int P(\alpha) \sum_n \sum_m c_{nm} \alpha^m \alpha^n \, d^2 \alpha
\]
[III-85a]

or, finally, the \text{“optical equivalence” theorem}
\[
\left\langle f^{(N)}(a, a^\dagger) \right\rangle = \int \int P(\alpha) f^{(N)}(\alpha, \alpha^*)
\]
[III-85b]

j. \textbf{The Uncertainty Relationship for } \{q, \pi\}:

Since \( [a, a^\dagger] = 1 \) we see from Equation \[ \text{III-64a} \] that
\[
\left\langle \Delta q^2 \right\rangle = \langle q^2 \rangle - \langle q \rangle^2
\]
\[
= \left\langle a^\dagger a^\dagger \right\rangle \exp \left[ 2 i \gamma \right] + \left\langle a a \right\rangle \exp \left[ -2 i \gamma \right] + \left\langle a^\dagger a \right\rangle + \left\langle a a^\dagger \right\rangle
\]
\[
- \left\langle a^\dagger a^\dagger \right\rangle \exp \left[ 2 i \gamma \right] - \left\langle a a \right\rangle \exp \left[ -2 i \gamma \right] - 2 \left\langle a^\dagger a \right\rangle \langle a \rangle
\]
[III-86]
\[
= \left\langle \Delta q^2 \right\rangle + 1
\]

where \left\langle A \right\rangle \text{ symbollizes the normally ordered expectation value of the operator } A. \text{ From Equation [III-85b]}
\[
\left\langle \Delta q^2 \left( a, a^\dagger \right) \right\rangle = \int \int P(\alpha) \Delta q^2(\alpha, \alpha^*) d^2 \alpha
\]
[III-87]
\[
\langle : \Delta \theta^2 \left( a, a^{\dagger} \right) : \rangle = \int \int P(\alpha) \left[ \Delta \alpha^* \exp(i \gamma) + \Delta \alpha \exp(-i \gamma) \right]^2 d^2 \alpha \quad \text{[III-88]}
\]

If we choose \( \gamma \) (and \( P(\alpha) \)) such that \( \langle : \Delta \theta^2 \left( a, a^{\dagger} \right) : \rangle < 0 \), then \( \mathbb{E} \{ \Delta \theta^2 \} > 1 \) and \( \mathbb{E} \{ \Delta \pi^2 \} > 1 \) (squeezed states)!