

VIA. ON CLASSICAL OPTICAL INTERFERENCE, COHERENCE AND FLUCTUATIONS

Light from a real physical source is never, strictly speaking, monochromatic nor does it emanate from a single point in space. Both the amplitude and the phase of the wave field generated by a real source undergoes irregular fluctuations. Within a time interval of the order of the putative *coherence time* the amplitude and phase of the radiation remain relatively stable and, thus, it may be treated as monochromatic. For time intervals long compared to the coherence time, the effects of fluctuations become manifest. In particular, interference phenomena, the hallmark of wave behavior, are observed only when the interfering beams traverse paths that differ in length by an amount small compared to the *coherence length* -- *i.e.* the product of the coherence time and the velocity of light. In this first section we review the classical analysis and interpretation of interference effects in terms of the notions of *partial coherence*.

COMPLEX REPRESENTATION OF POLYCHROMATIC FIELDS:

Let us first carefully define the mathematical framework of the subject. We assume that the observable optical field may be represented by a Fourier integral transform ¹

$$E^{(r)}(t) = \int E(\omega) \exp[-i\omega t] d\omega \quad [\text{VIA-1}]$$

Since of necessity $E^{(r)}(t)$ must be real

¹ For this assumption to be valid, it is sufficient that the function $E(t)$ be *square-integrable* -- *i.e.*

$$\int [E^{(r)}(t)]^2 dt < \infty$$

$$E(-) = E() \quad [\text{VIA-2}]$$

and, consequently, Equation [VIA-1] may be re-expressed in terms of an integral over positive frequencies -- viz.

$$E^{(r)}(t) = \int_0 \{ E() \exp[-i t] + c.c. \} d \quad [\text{VIA-3}]$$

Further, if we write

$$E() = \frac{1}{2} a() \exp[i ()] , \quad [\text{VIA-4}]$$

where $a()$ and $()$ are both real, then the observable optical field may be expressed in the form

$$E^{(r)}(t) = \int_0 a() \cos\{ i [() - t] \} d \quad [\text{VIA-5}]$$

which is a strict generalization of the complex or *phasor* representation of real monochromatic fields. To complete the generalization, we follow Dennis Gabor² and introduce the so called complex *analytic signal*

$$E(t) = \int_0 a() \exp\{ i [() - t] \} d = 2 \int_0 E() \exp[-i t] d \quad [\text{VIA-6}]$$

which is to be associated with the real observable field -- viz. $E^{(r)}(t) = \mathbf{Re}[E(t)]$. As we shall see, this rather careful, or should we say pedantic, introduction of the analytic signal

² D. Gabor, *J. Inst. Elec. Engrs.* (London), **93**, 2529 (1946).

greatly facilitates and validates the use of complex quantities in handling *cycle-averaged* properties -- e.g. energy and intensity -- of polychromatic optical fields.³ In most applications of interest in optics, the spectral amplitudes will have appreciable values only within a spectral width $\Delta\omega$ which is small compared to some mean frequency $\langle \omega \rangle$. Thus, it is often useful to express the *analytic signal* in terms of a temporal modulation of a mean monochromatic -- viz.

$$E(t) = a(t) \exp \left\{ i \left[\langle \omega \rangle t - \phi(t) \right] \right\} \quad [\text{VIA-7}]$$

Using Equation [VIA-6] we see that

$$a(t) \exp \left[i \langle \omega \rangle t \right] = 2 \int_0^\infty E(\omega) \exp \left[-i (\omega - \langle \omega \rangle) t \right] d\omega = \tilde{E}(\omega - \langle \omega \rangle) \exp \left[-i \langle \omega \rangle t \right] \quad [\text{VIA-8}]$$

where $\tilde{E}(\omega) = 2 \int_0^\infty E(\omega + \langle \omega \rangle) \exp \left[-i \omega t \right] dt$: by assumption, $\tilde{E}(\omega)$ will be appreciable only near $\omega = \langle \omega \rangle$.

The following list of transforms are included for reference:

$$a(t) = \sqrt{ \left[E^{(r)}(t) \right]^2 + \left[E^{(i)}(t) \right]^2 } = \sqrt{ E(t) E^*(t) } = |E(t)| \quad [\text{VIA-9a}]$$

³ The following identities involving *improper functions* are invaluable when treating analytic signals -- see Beran and Parrent:

$$\frac{1}{2} \int_{-\infty}^{\infty} \exp \left(i x (y - y) \right) dx = \delta(y - y)$$

$$\frac{1}{2} \int_0^\infty \exp \left[\mp i x (y - y) \right] dx = \pm \delta(y - y) = \frac{1}{2} \int_{-\infty}^{\infty} \exp \left[\mp i x (y - y) \right] dx = \pm \frac{i}{y - y}$$

$$E^{(i)}(t) = \langle E \rangle + \tan^{-1} \frac{E^{(i)}(t)}{E^{(r)}(t)} = \langle E \rangle + \tan^{-1} i \frac{E^{(i)}(t) - E^{(r)}(t)}{E^{(i)}(t) + E^{(r)}(t)} \quad [\text{VIA-9b}]$$

$$E^{(r)}(t) = a(t) \cos\left\{ i \left[\langle E \rangle t - \langle E \rangle t \right] \right\} \quad [\text{VIA-9c}]$$

$$E^{(i)}(t) = a(t) \sin\left\{ i \left[\langle E \rangle t - \langle E \rangle t \right] \right\}. \quad [\text{VIA-9d}]$$

Using Equations [VIA-1] and [VIA-6], we may easily establish a form of Parseval's formula

$$\begin{aligned} \int_{-\infty}^{\infty} [E^{(r)}(t)]^2 dt &= \int_{-\infty}^{\infty} [E^{(i)}(t)]^2 dt = \frac{1}{2} \int_{-\infty}^{\infty} |E(t)|^2 dt \\ &= \int_{-\infty}^{\infty} E(t) E(t) \exp[-i(\langle E \rangle + \langle E \rangle)t] dt \quad [\text{VIA-10}] \\ &= 2 \int_{-\infty}^{\infty} |E(t)|^2 dt = 4 \int_{-\infty}^{\infty} |E(t)|^2 dt \end{aligned}$$

We have thus far assumed that field is defined for all values of t. In practical terms, the field or, more likely, the observation of the field is defined only within some finite time interval $-T \leq t \leq T$. In all instances of any consequence, T is much larger than the physically significant times $2/\langle E \rangle$ and $2/\langle \dot{E} \rangle$ so that we may feel some confident in taking the idealization $T \rightarrow \infty$. However, *stationary*⁴ requires that the time averaged energy of the field, which is proportional to

⁴ A random process is characterized as *stationary* when all of the moments of the random variables are independent of **absolute time**.

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [E^{(r)}(t)]^2 dt, \quad [\text{VIA-11}]$$

must be finite and, thus, a stationary field function is not square-integrable! The problem of analyzing such functions has been a much discussed issue in the mathematics literature.⁵ However, as applied physicists, we brush these mathematical niceties under the rug in what follows. We proceed by firmly asserting that optical fields of interest do, indeed, have Fourier transforms and chance the consequences of *truncation errors*.

The cycled-averaged intensity or power spectral density is a crucial element in the analysis of coherence effects and, from Equation [VIA-6], it is proportional

$$\begin{aligned} |E(\omega)|^2 &= \frac{1}{4} \int_{-T}^T \int_{-T}^T \exp[i\omega(t-t')] E(t)E(t') dt dt' \\ &= \frac{1}{4} \int_{-T}^T \int_{-T}^T \exp[i\omega(t-t')] E(t+\tau)E(t) dt d\tau \end{aligned} \quad [\text{VIA-12}]$$

Consideration of this expression leads us to a logical definition of the *first-order temporal correlation function* of the electric field as

$$G^{(1)}(\tau) = \langle E(t+\tau)E(t) \rangle = \frac{1}{2T} \int_{-T}^T E(t+\tau)E(t) dt \quad [\text{VIA-13}]$$

So that Equation [VIA-12] becomes

⁵ See, in particular, N. Wiener, *Acta. Math.*, **55**, 117 (1930).

$$|E(\tau)|^2 = \frac{T}{8} \langle E(t+\tau)E(t) \rangle \exp[i\omega\tau] d = \frac{T}{8} \langle E(t)E(t+\tau) \rangle \exp[-i\omega\tau] d \quad [\text{VIA-14}]$$

and the normalized power spectral density (distribution) is defined as

$$\begin{aligned} F(\omega) &= \frac{1}{T} \int_{-T/2}^{T/2} |E(\tau)|^2 \exp[i\omega\tau] d\tau \\ &= \frac{1}{2T} \int_{-T/2}^{T/2} \langle E(t+\tau)E(t) \rangle \exp[i\omega\tau] d\tau \langle |E(t)|^2 \rangle^{-1} \quad [\text{VIA-15}] \\ &= \frac{1}{2T} \langle E(t+\tau)E(t) \rangle \exp[i\omega\tau] d\tau \end{aligned}$$

where

$$\langle E(t+\tau)E(t) \rangle = \frac{\langle E(t+\tau)E(t) \rangle}{\langle E(t)E(t) \rangle} \quad [\text{VIA-16}]$$

This normalized correlation function is usually called the field's *complex degree of first-order temporal coherence*.⁶

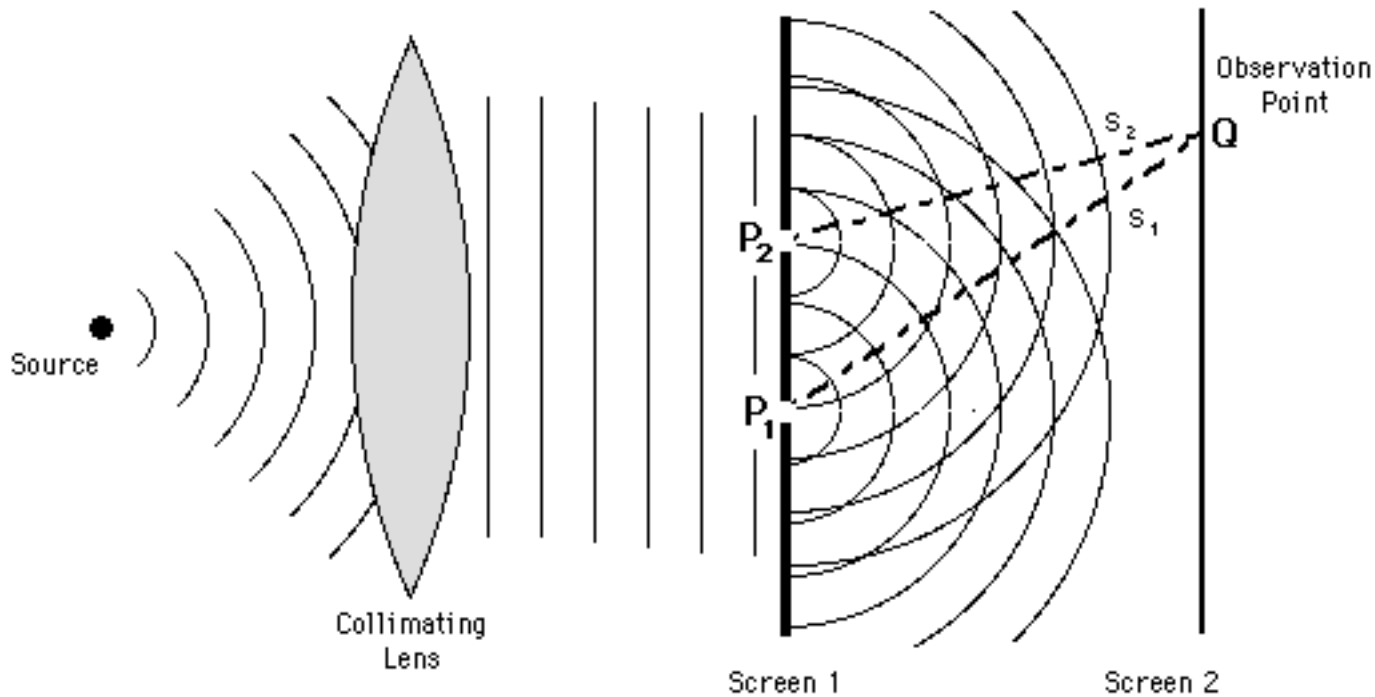
YOUNG'S INTERFERENCE EXPERIMENT -- A RUDIMENTARY MEASUREMENT OF TEMPORAL COHERENCE

By considering in some detail the simplest kind of experiment in which coherence effects are important -- *i.e.* Young's (or Michelson's) observation of interference fringes -- we can

⁶ It is easy to show that

$$F(\omega) = \frac{1}{2T} \text{Re} \int_{-T/2}^{T/2} \langle E(t+\tau)E(t) \rangle \exp[i\omega\tau] d\tau$$

identify the significance of a light source's degree of first-order temporal coherence and demonstrate how that coherence can be measured. Consider the following basic experimental configuration:



The time integrated/averaged intensity at a given point \mathbf{P} is proportional to

$$I(P) = 2 \left\langle \left[E^{(r)}(P, t) \right]^2 \right\rangle = \langle E(P, t) E(P, t) \rangle. \quad [\text{VIA-17}]$$

The real instantaneous field the point \mathbf{Q} is derivable from the following analytic signal:

$$\begin{aligned} E(Q, t) &= C_1 E(P_1, t - t_1) + C_2 E(P_2, t - t_2) \\ &= C_1 E(P_1, t - s_1/c) + C_2 E(P_2, t - s_2/c) \end{aligned} \quad [\text{VIA-18}]$$

It follows that the time integrated/averaged intensity at the point \mathbf{Q} is proportional to

$$I(Q) = C_1 C_1 \langle E(P_1, t - t_1) E(P_1, t - t_1) \rangle + C_1 C_2 \langle E(P_1, t - t_1) E(P_2, t - t_2) \rangle + C_2 C_1 \langle E(P_2, t - t_2) E(P_1, t - t_1) \rangle + C_2 C_2 \langle E(P_2, t - t_2) E(P_2, t - t_2) \rangle \quad [\text{VIA-19}]$$

If the field is *stationary*, we may with confidence shift the time origin in the various expressions so that

$$\begin{aligned} \langle E(P_n, t - t_n) E(P_n, t - t_n) \rangle &= \langle E(P_n, t) E(P_n, t) \rangle \\ &= I(P_n) = {}^{(i)}_{nm}(0) \end{aligned} \quad [\text{VIA-20a}]$$

$$\begin{aligned} \langle E(P_n, t - t_n) E(P_m, t - t_m) \rangle &= \langle E(P_n, t + t_m - t_n) E(P_m, t) \rangle \\ &= {}^{(i)}_{nm}(t_m - t_n) = {}^{(i)}_{nm} \frac{s_m - s_n}{c} \end{aligned} \quad [\text{VIA-20b}]$$

The normalized coherence function

$${}^{(i)}_{12}(\) = \frac{{}^{(i)}_{12}(\)}{\sqrt{{}^{(i)}_{11}} \sqrt{{}^{(i)}_{22}}(0)} = \frac{{}^{(i)}_{12}(\)}{\sqrt{I(P_1)} \sqrt{I(P_2)}} \quad [\text{VIA-21}]$$

is often called the *complex mutual degree of coherence*. Finally, we write the *general interference law for stationary optical fields* as

$$I(Q) = I_1(Q) + I_2(Q) + 2 \sqrt{I_1(Q)} \sqrt{I_2(Q)} {}^{(i)}_{12} \frac{s_2 - s_1}{c} \quad [\text{VIA-21a}]$$

where $I_1(Q) = |C_1|^2 I(P_1)$ and $I_2(Q) = |C_2|^2 I(P_2)$.

Thus the degree of first-order temporal coherence may be expressed directly in terms of well-defined experimentally observable quantities -- viz.

$${}_{12}^{(i)} \frac{s_2 - s_1}{c} = \frac{I(Q) - I_1(Q) - I_2(Q)}{2\sqrt{I_1(Q)}\sqrt{I_2(Q)}} \quad [\text{VIA-21b}]$$

where $I(Q)$ and $I_n(Q)$ are, respectively, proportional to the intensity measured at Q with both pinholes open or with only the n th pinhole open. The **visibility of the fringes**

given by

$$\mathcal{V}(Q) = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1(Q)}\sqrt{I_2(Q)}}{I_1(Q) + I_2(Q)} \left| {}_{12}^{(i)} \frac{s_2 - s_1}{c} \right| \quad [\text{VIA-22}]$$

MODELS OF CHAOTIC LIGHT SOURCES

A COLLISION-BROADENED SOURCE -- AN EXAMPLE OF HOMOGENEOUS-BROADENING:

Perhaps, the simplest model of a *chaotic source* is that of a collision interrupted oscillator. In this model one pictures an ensemble of *atomic* radiators each of which emits a field of constant amplitude, oscillating at a fixed frequency $\langle \omega \rangle$. The phase of the continuous output of each radiator is randomly modulated by the dephasing effect of collisions which occur random at some mean collision rate of ν^{-1} . The figures on the next page shows samples of the field radiated per atom (left-hand graphs) and the intensity radiated per atom (right-hand graphs) for ensembles of 50, 100, 200 and 300 collision interrupt atoms.⁷ This model is treated "experimentally" in the discussion entitled *A "Toy" Model of a Randomly Fluctuating Optical Field* which is located at

⁷ The figures show samples of the sums $\sum_{i=1}^N \sin \phi_i / N$ and $\sum_{i=1}^N \sin^2 \phi_i / N$ for N atoms with the phase radiated by each atom ϕ_i distributed uniformly between 0 and 2π .

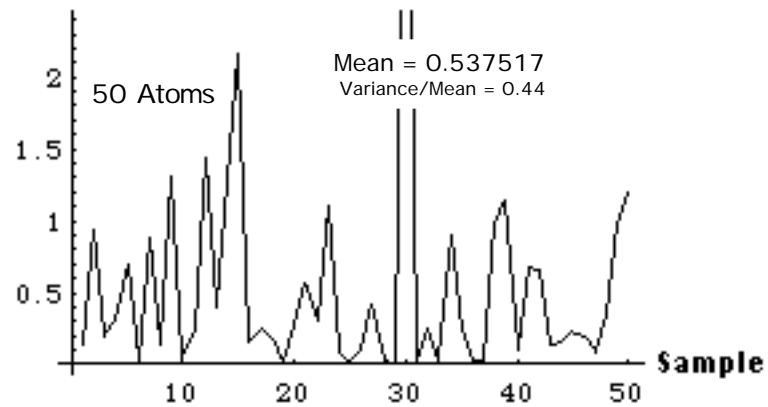
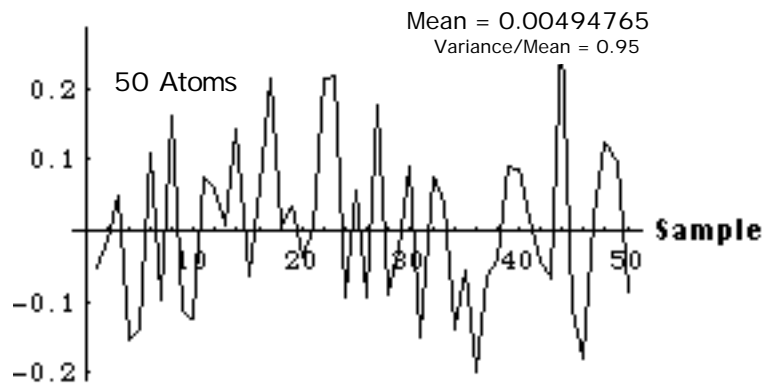
http://www.deas.harvard.edu/courses/ap216/lectures/ls_3/ls3_u6A/toy/toy.html. From that discussion, it is reasonable to write for a *collision-broadened* radiator

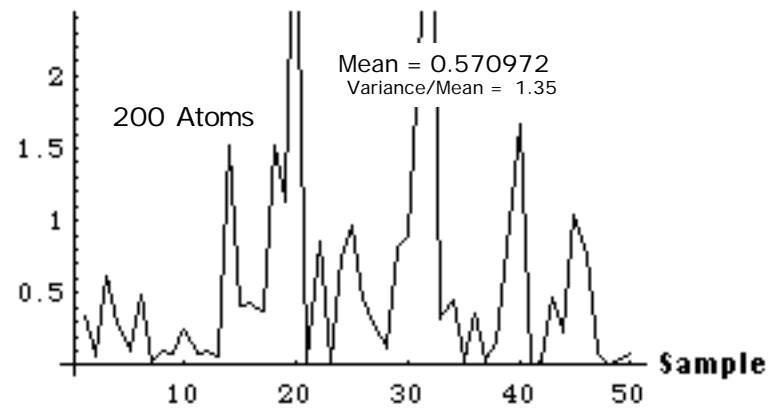
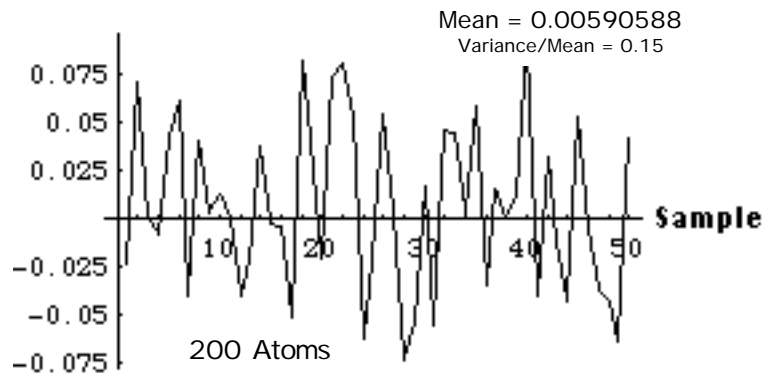
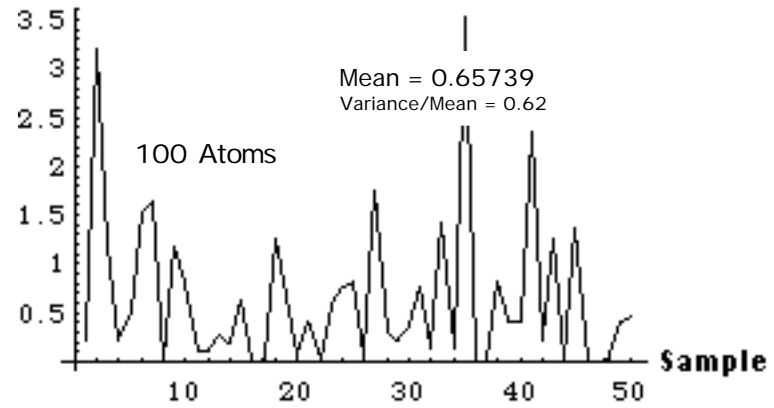
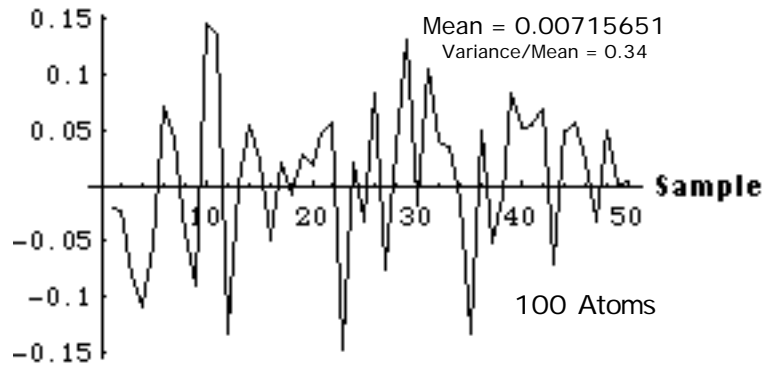
$$\langle \rho(t) \rangle = \exp[-i(\omega - \omega_0)t - \gamma t] \quad [\text{VIA-23}]$$

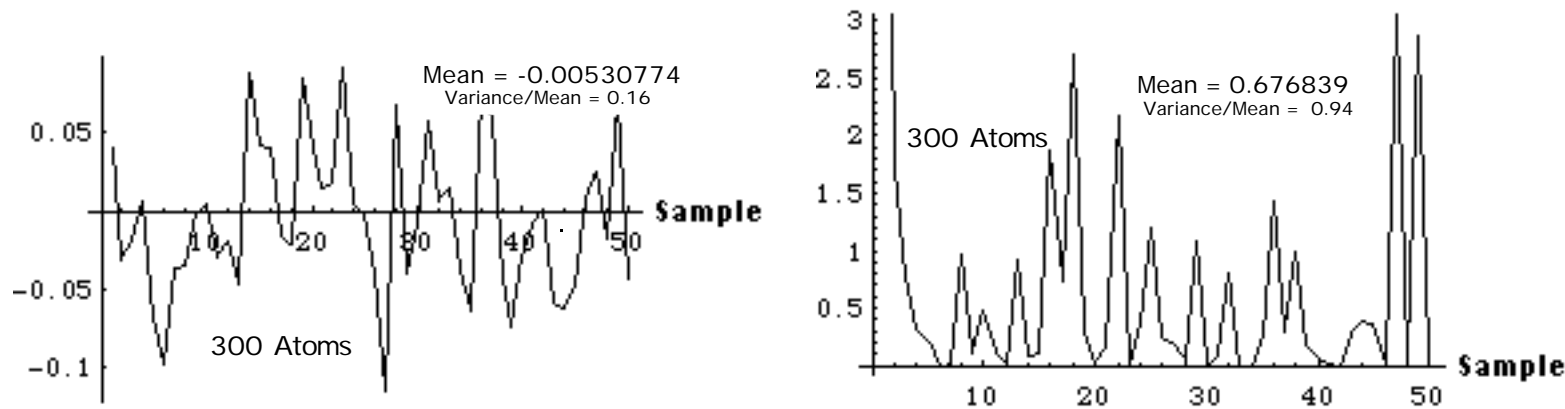
(see a graph of this function below). From Equation [VIA-15], we see that the spectrum of a collision-broadened source is given by the Lorentzian form

$$F(\omega) = \frac{1}{\gamma} \left(\frac{\gamma}{2} \right)^2 + (\omega - \omega_0)^2 \quad [\text{VIA-24}]$$

FIELD AND INTENSITY FLUCTUATIONS FOR COLLISION- OR DOPPLER-BROADENED SOURCES







DOPPLER-BROADENED SOURCES -- AN EXAMPLE OF INHOMOGENEOUS-BROADENING:

A rather more intricate model of a *chaotic source* is that of a **Doppler-broadening** system of oscillators. In this model one pictures an ensemble of **randomly moving atomic** radiators each of which emits a field of constant amplitude, oscillating at a fixed frequency, but the observed frequency of a given atom is Doppler-shifted with respect to that fixed frequency. In the Doppler model, we write the sum of fields emitted by a ensemble of moving radiators as

$$E(t) = E_0 \exp\left\{-i \left[\sum_i \omega_i t - \phi_i \right]\right\} \quad [\text{VIA-25}]$$

where the ω_i 's are the frequencies of individual atoms which are Doppler shifted from the mean radiated frequency $\langle \omega \rangle$. In this model, the effects of collisions are neglected and, consequently, the first-order correlation function is written as⁸

⁸ Since the dynamics of the individual *atoms* are uncorrelated, all cross-terms vanish.

$$\begin{aligned} \langle E(z_1 t_1) E(z_2 t_2) \rangle &= E_0^2 \sum_{i,j} \left\langle \exp \left\{ i \left[\omega_i (t_1 - z_1/c) + \omega_j (t_2 - z_2/c) - \phi_j \right] \right\} \right\rangle \\ &= E_0^2 \sum_i \left\langle \exp \left[-i \omega_i \tau \right] \right\rangle \end{aligned} \quad \text{[VIA-26]}$$

where $\tau = t_2 - z_2/c - t_1 + z_1/c$. The sum may be converted to an integral over a Gaussian distribution of Doppler-shifted frequencies and, hence,

$$\begin{aligned} \langle E(z_1 t_1) E(z_2 t_2) \rangle &= E_0^2 \int_0^\infty \exp \left[-i \omega \tau \right] \exp \left(-\frac{(\omega - \langle \omega \rangle)^2}{2 \sigma^2} \right) d\omega \\ &= N_{\text{atom}} E_0^2 \exp \left[-i \langle \omega \rangle \tau - \sigma^2 \tau^2 / 2 \right] \end{aligned} \quad \text{[VIA-27]}$$

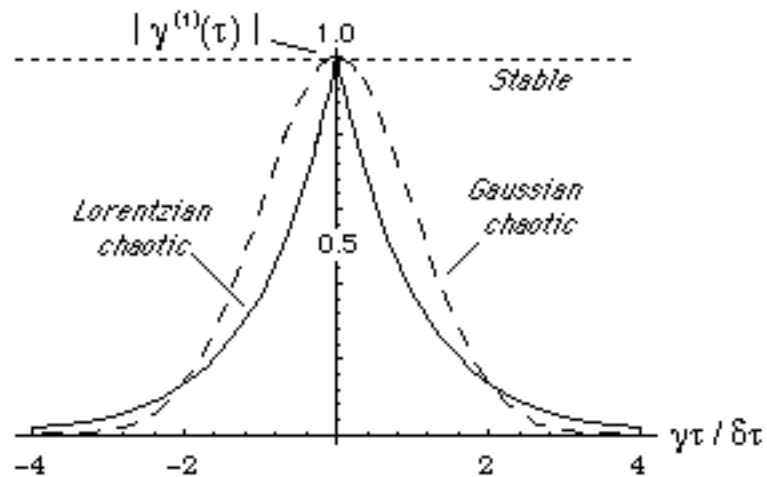
The corresponding degree of first-order temporal coherence is (see a graph of this function below)

$$\langle \hat{E}(\tau) \rangle = \exp \left[-i \langle \omega \rangle \tau - \sigma^2 \tau^2 / 2 \right] \quad \text{[VIA-28]}$$

and the associated normalized power spectral density follows from Equation [VIA-15] as

$$F(\omega) = \left[2 \sigma^2 \right]^{-1/2} \exp \left(-\frac{(\omega - \langle \omega \rangle)^2}{2 \sigma^2} \right) \quad \text{[VIA-29]}$$

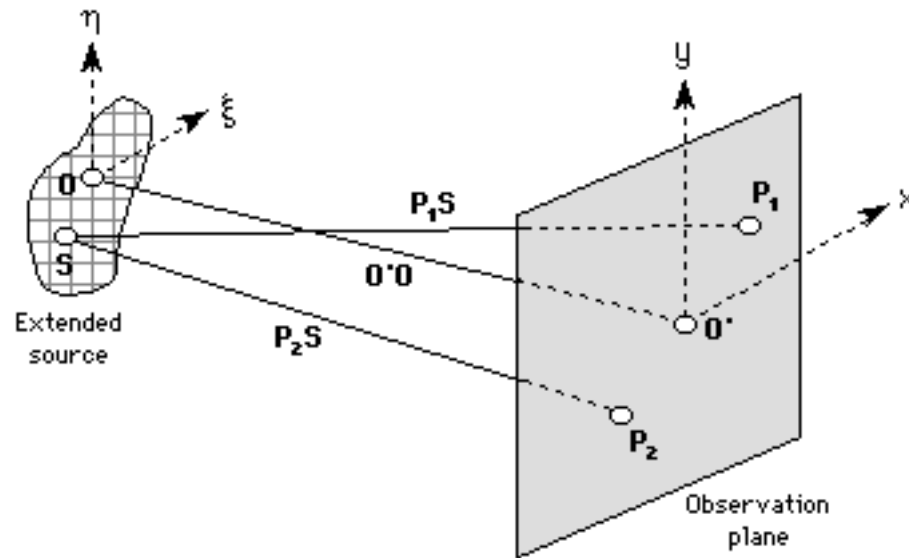
DEGREE OF FIRST-ORDER COHERENCE - ABSOLUTE VALUE



SPATIAL COHERENCE: THE VAN CITTERT-ZERNIKE

In the rudimentary treatment of the Young interference experiment recapitulated above, we assume an ideal point source and, thus, neglect any effects of *spatial coherence*. In particular, we assume that the field at points \mathbf{P}_1 and \mathbf{P}_2 due to a given radiator are in time synchronism. However, when we deal with extended sources, we must enlarge our notion of coherence. The subject of spatial coherence was first developed by van Cittert and, later, it was extended by Zernike.⁹ To define the problem, let us consider the following geometry:

⁹ See P. H. van Cittert, *Physica*, **1**, 201 (1934) and F. Zernike, *Physica*, **5**, 785 (1938).



The task before us is to determine the so called mutual intensity $J(P_1, P_2)$ due to the extended source which might be observed for the two points located on the observation plane. For simplicity, we assume that extended source is in a plane parallel to the observation plane and that it can be divided into cells of statistically independent radiators. Thus, the total field at any point on the observation plane is given by summing the fields due to each of the cells -- viz.

$$E(P, t) = \sum_k E_k(P, t) . \quad [\text{VIA-30}]$$

The mutual intensity is then defined as

$$J(P_1, P_2) = \langle E(P_1, t) E(P_2, t) \rangle = \langle E_k(P_1, t) E_k(P_2, t) \rangle + \langle E_k(P_1, t) E_{k'}(P_2, t) \rangle . \quad [\text{VIA-31a}]$$

Because of the assumed statistical independence of the cells, the cross-terms vanish so that

$$J(P_1, P_2) = \left\langle E_k(P_1, t) E_k(P_2, t) \right\rangle_k . \quad [\text{VIA-31b}]$$

In the spirit of the *Huygens principle*, we assert that component fields radiated by each coherent cell are spherical waves of the form

$$E_k(P, t) = E(S_k, t - \frac{P S_k}{c}) \frac{\exp[-i \langle \rangle t - \frac{P S_k}{c}]}{P S_k} . \quad [\text{VIA-32}]$$

where $E(S_k, t)$ is field radiated by each element of the source and $P S_k$ is, obviously, the distance between the source and observation positions. Therefore, we can write

$$\begin{aligned} \left\langle E_k(P_1, t) E_k(P_2, t) \right\rangle &= \left\langle E(S_k, t - \frac{P_1 S_k}{c}) E(S_k, t - \frac{P_2 S_k}{c}) \right\rangle \frac{\exp[i \langle \rangle (P_1 S_k - P_2 S_k)/c]}{[P_1 S_k] [P_2 S_k]} \\ &= \left\langle E(S_k, t) E(S_k, t - (P_2 S_k - P_1 S_k)/c) \right\rangle \frac{\exp[i \langle \rangle (P_1 S_k - P_2 S_k)/c]}{[P_1 S_k] [P_2 S_k]} \end{aligned} \quad [\text{VIA-33}]$$

In most instances, the path difference will be small compared to the coherence length of the cell, so that, in the limit of many small cells, we obtain the famous *van Cittert-Zernike theorem* in the form

$$J(P_1, P_2) = \int_{\text{Source}} I(S) \frac{\exp[i \langle k \rangle (P_1 S - P_2 S)]}{[P_1 S] [P_2 S]} dA_S \quad [\text{VIA-34}]$$

and the first-order degree of mutual (spatial) coherence is defined as

$$\begin{aligned}
 I_{12} &= I(P_1, P_2) = \frac{J(P_1, P_2)}{\sqrt{I(P_1)} \sqrt{I(P_2)}} \\
 &= \frac{1}{\sqrt{I(P_1)} \sqrt{I(P_2)}} \int_{\text{Source}} I(S) \frac{\exp [i \langle k \rangle (P_1 S - P_2 S)]}{[P_1 S] [P_2 S]} dA_S
 \end{aligned}
 \tag{VIA-35a}$$

where

$$I(P_n) = \int_{\text{Source}} \frac{I(S)}{[P_n S]^2} dA_S .
 \tag{VIA-35b}$$

For most problems of interest, we may take

$$P_n S = \sqrt{R^2 + (x_n -)^2 + (y_n -)^2} \approx R + \frac{(x_n -)^2 + (y_n -)^2}{2R}
 \tag{VIA-36}$$

and, hence, we may approximate Equation [VIA-35a] as

$$\begin{aligned}
 I_{12} &= \frac{\int_{\text{Source}} I(S) \exp [i \langle k \rangle \frac{(x_1 -)^2 - (x_2 -)^2 + (y_1 -)^2 - (y_2 -)^2}{2R}] dA_S}{\int_{\text{Source}} I(S) dA_S}
 \end{aligned}
 \tag{VIA-37a}$$

Further, we see that in the **Fraunhofer or far-field approximation**, the mutual coherence is the Fourier transform of the source intensity distribution -- viz.

$$I(\vec{r}_1, \vec{r}_2) = \frac{\int_{\text{Source}} \int_{\text{Source}} \exp[i\langle k \rangle (x_1 - x_2)/R] \exp[i\langle k \rangle (y_1 - y_2)/R] d^2x_1 d^2x_2}{\int_{\text{Source}} \int_{\text{Source}} d^2x_1 d^2x_2} \quad ! \quad [\text{VIA-37b}]$$

-- and, consequently, **the measurement of the degree of mutual coherence of a remote source provides a means to measure the angular size of that extended sources -- i.e. stellar objects.**

We see then that a complete first-order description of the coherence properties of a classical field is embodied in the **complex first-order degree of spatial-temporal coherence** which is defined as

$$\gamma(\vec{r}_1, t_1; \vec{r}_2, t_2) = \frac{\langle E(\vec{r}_1, t_1) E(\vec{r}_2, t_2) \rangle}{\sqrt{\langle |E(\vec{r}_1, t_1)|^2 \rangle \langle |E(\vec{r}_2, t_2)|^2 \rangle}} \quad [\text{VIA-38a}]$$

where

$$\langle E(\vec{r}_1, t_1) E(\vec{r}_2, t_2) \rangle = \frac{1}{2T} \int_{-T}^T E(\vec{r}_1, t_1 + t) E(\vec{r}_2, t_2 + t) dt. \quad [\text{VIA-38b}]$$

FREE SPACE PROPAGATION OF COHERENCE FUNCTIONS:

We derive here an equation which allows us to generalize the notions implicit in the van Cittert-Zernike theorem by providing us a description of how coherence propagates through space. We start by assuming that the *analytic signal* representing the field of interest satisfies an wave equation of the form

$${}^2 E(\vec{\mathbf{r}}_n, t_n) = \frac{1}{c^2} \frac{{}^2}{t_n^2} E(\vec{\mathbf{r}}_n, t_n) \quad [\text{VIA-39}]$$

If we multiply through by $E(\vec{\mathbf{r}}_2, t_2)$, we see that

$$\begin{aligned} {}^2_1 (\vec{\mathbf{r}}_1 t_1, \vec{\mathbf{r}}_2 t_2) &= {}^2_1 \langle E(\vec{\mathbf{r}}_1, t_1) E(\vec{\mathbf{r}}_2, t_2) \rangle \\ &= \frac{1}{c^2} \left\langle \frac{{}^2}{t_1^2} E(\vec{\mathbf{r}}_1, t_1) E(\vec{\mathbf{r}}_2, t_2) \right\rangle. \quad [\text{VIA-40a}] \\ &= \frac{1}{c^2} \frac{{}^2}{t_1^2} (\vec{\mathbf{r}}_1 t_1, \vec{\mathbf{r}}_2 t_2) \end{aligned}$$

Similarly

$${}^2_2 (\vec{\mathbf{r}}_1 t_1, \vec{\mathbf{r}}_2 t_2) = \frac{1}{c^2} \frac{{}^2}{t_2^2} (\vec{\mathbf{r}}_1 t_1, \vec{\mathbf{r}}_2 t_2). \quad [\text{VIA-40b}]$$

Therefore the coherence function must satisfy the fourth-order equation

$${}^2_1 {}^2_2 (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2;) = \frac{1}{c^4} \frac{{}^2}{t_1^2 t_2^2} (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2;) \quad [\text{VIA-41}]$$

The implication of Equations [VIA-40] and [VIA-41] is that **coherence is a field that propagates through space** and may be treated by methods which apply to other field variables. In fact, the content of the van Cittert-Zernike theorem is a description of how the coherence of extended source propagates to remote observation points. This point of view is particularly valuable in the treatment of wave propagation through random media. As a beam propagates through such a medium, the effect of random scattering events scrambles the phase of the beam and the simple notion of a field rapidly loses meaning. However, the evolution of the coherence function provides an important measure of the statistical properties of the random medium.

HIGHER-ORDER CORRELATION FUNCTIONS -- A CLASSICAL EXPLICATION OF THE FAMOUS HANBURY BROWN-TWISS EXPERIMENT:

Generalizing Equation [VIA-38a], the *degree of nth-order spatial-temporal coherence* can be defined as¹⁰

$$^{(n)} (\vec{r}_1, t_1; \dots; \vec{r}_n, t_n; \vec{r}_{n+1}, t_{n+1}; \dots; \vec{r}_{2n}, t_{2n}) = \frac{\langle E(\vec{r}_1, t_1) \dots E(\vec{r}_n, t_n) E(\vec{r}_{n+1}, t_{n+1}) \dots E(\vec{r}_{2n}, t_{2n}) \rangle}{\sqrt{\langle |E(\vec{r}_1, t_1)|^2 \rangle \dots \langle |E(\vec{r}_n, t_n)|^2 \rangle \langle |E(\vec{r}_{n+1}, t_{n+1})|^2 \rangle \dots \langle |E(\vec{r}_{2n}, t_{2n})|^2 \rangle}} \quad [\text{VIA-42}]$$

In particular, the *degree of second-order temporal coherence* is defined as

$$^{(2)} (\vec{r}_{\text{ref}},) = \frac{\langle |E(\vec{r}_{\text{ref}}, t)|^2 |E(\vec{r}_{\text{ref}}, t +)|^2 \rangle}{\langle |E(\vec{r}_{\text{ref}}, t)|^2 \rangle^2} = \frac{\langle \bar{I}(\vec{r}_{\text{ref}}, t) \bar{I}(\vec{r}_{\text{ref}}, t +) \rangle}{\langle \bar{I}(\vec{r}_{\text{ref}}, t) \rangle^2} \quad [\text{VIA-43}]$$

We shall see that this function is an important measure of the relative timing of intensity fluctuations.¹¹ As a first step, let first find an expression for the average intensity radiated by a collection of independent oscillators -- viz.

¹⁰ R. J. Glauber, in *Quantum Optics and Electronics, Les Houches, 1964* (edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji), p63, Gordon and Breach (1965).

¹¹ **By the famous Schwartz inequality**

$$\left| \langle a^* b \rangle \right|^2 \leq \langle |a|^2 \rangle \langle |b|^2 \rangle$$

so that $\langle \bar{I}(\vec{r}_{\text{ref}}, t) \bar{I}(\vec{r}_{\text{ref}}, t +) \rangle \leq \langle \bar{I}(\vec{r}_{\text{ref}}, t) \rangle \langle \bar{I}(\vec{r}_{\text{ref}}, t +) \rangle$

and quite generally $^{(2)} (\vec{r}_{\text{ref}},) \leq 1$

$$\begin{aligned} \bar{I} \quad \langle \bar{I}(t) \rangle &= [c/2] \langle |E(t)|^2 \rangle = [c/2] \left\langle \sum_{i,j} E_i(t) E_j(t) \right\rangle \\ &= [c/2] \sum_i \langle |E_i(t)|^2 \rangle \end{aligned} \quad [\text{VIA-44}]$$

In this expression all of the cross-terms are presumed to vanish, since the radiation from any given atom is statistically uncorrelated with that of any other atom. For the collision- and Doppler-broadening models discussed above, the oscillators differ only in phase and this average can be expressed as

$$\bar{I} \quad \langle \bar{I}(t) \rangle = N_{\text{atom}} [c/2] |a|^2. \quad [\text{VIA-45}]$$

To obtain the root-mean-square deviation in the cycle average of the intensity, we first calculate

$$\begin{aligned} \langle [\bar{I}(t)]^2 \rangle &= [c/2]^2 \langle |E(t)|^4 \rangle = [c/2]^2 \left\langle \sum_{i,j,k,l} E_i(t) E_j(t) E_k(t) E_l(t) \right\rangle \\ &= [c/2]^2 \sum_i \langle |E_i(t)|^4 \rangle + \sum_{i \neq j} \langle |E_i(t)|^2 |E_j(t)|^2 \rangle \\ &= [c/2]^2 \sum_i \langle |E_i(t)|^4 \rangle + \sum_{i \neq j} \langle |E_i(t)|^2 \rangle \langle |E_j(t)|^2 \rangle \\ &= [c/2]^2 \left(N_{\text{atom}} \langle |E_i(t)|^4 \rangle + 2 N_{\text{atom}} (N_{\text{atom}} - 1) \langle |E_i(t)|^2 \rangle^2 \right) \end{aligned} \quad [\text{VIA-46}]$$

Again, for the collision- and Doppler-broadening models, this average can be expressed as

$$\frac{\langle [\bar{I}(t)]^2 \rangle}{2 \langle \bar{I}(t) \rangle^2} = [N_{\text{atom}}]^2 [c/2]^2 |a|^4 [2 - (N_{\text{atom}})^{-1}] \quad \text{[VIA-47]}$$

and, thus, we find to a very good approximation that

$$\sqrt{\langle [\bar{I}(t)]^2 \rangle} - \langle \bar{I}(t) \rangle^2 \approx \bar{I} \quad \text{[VIA-48]}$$

which is consistent with the *experimental results* presented above. Let us now look at the **two-time average of the intensity** from a collection of statistically independent oscillators -- viz.

$$\begin{aligned} \langle \bar{I}(t_1) \bar{I}(t_2) \rangle &= [c/2]^2 \langle |E(t_1)|^2 |E(t_2)|^2 \rangle \\ &= [c/2]^2 \left\langle \sum_{i,j,k,l} E_i(t_1) E_j(t_2) E_k(t_1) E_l(t_2) \right\rangle \end{aligned} \quad \text{[VIA-49]}$$

As a result of statistical independence, most of the cross-terms vanish so that

$$\begin{aligned} \langle \bar{I}(t_1) \bar{I}(t_2) \rangle &= [c/2]^2 \left\langle |E_i(t_1)|^2 |E_i(t_2)|^2 \right\rangle \\ &+ \sum_{i,j} \left\langle |E_i(t_1)|^2 |E_j(t_2)|^2 \right\rangle + \sum_{i,j} \langle E_i(t_1) E_i(t_2) E_j(t_1) E_j(t_2) \rangle \end{aligned} \quad \text{[VIA-50]}$$

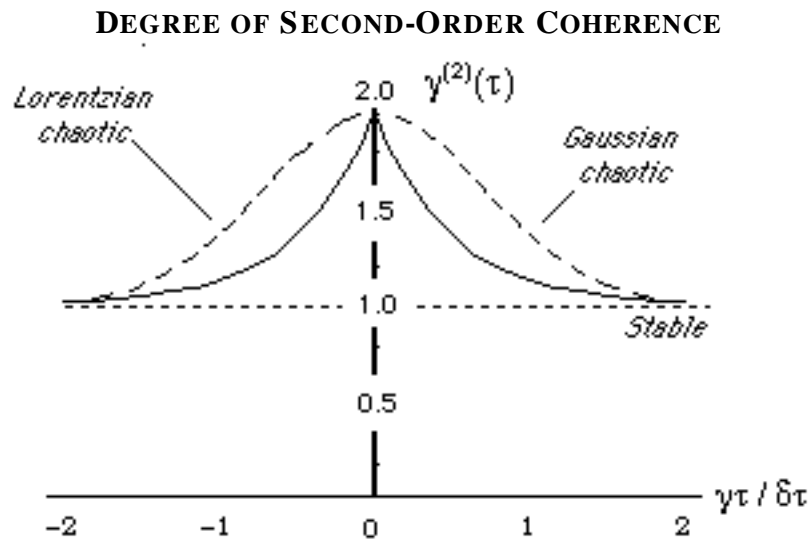
Again, for the collision- and Doppler-broadening models

$$\langle \bar{I}(t_1) \bar{I}(t_2) \rangle = \left[\frac{c}{2} \right]^2 N_{\text{text}} \left\langle |E_i(t_1)|^2 |E_i(t_2)|^2 \right\rangle + N_{\text{text}} (N_{\text{text}} - 1) \left\langle |E_i(t_1)|^2 \right\rangle^2 + \left| \langle E_i(t_1) E_i(t_2) \rangle \right|^2 \quad [\text{VIA-51}]$$

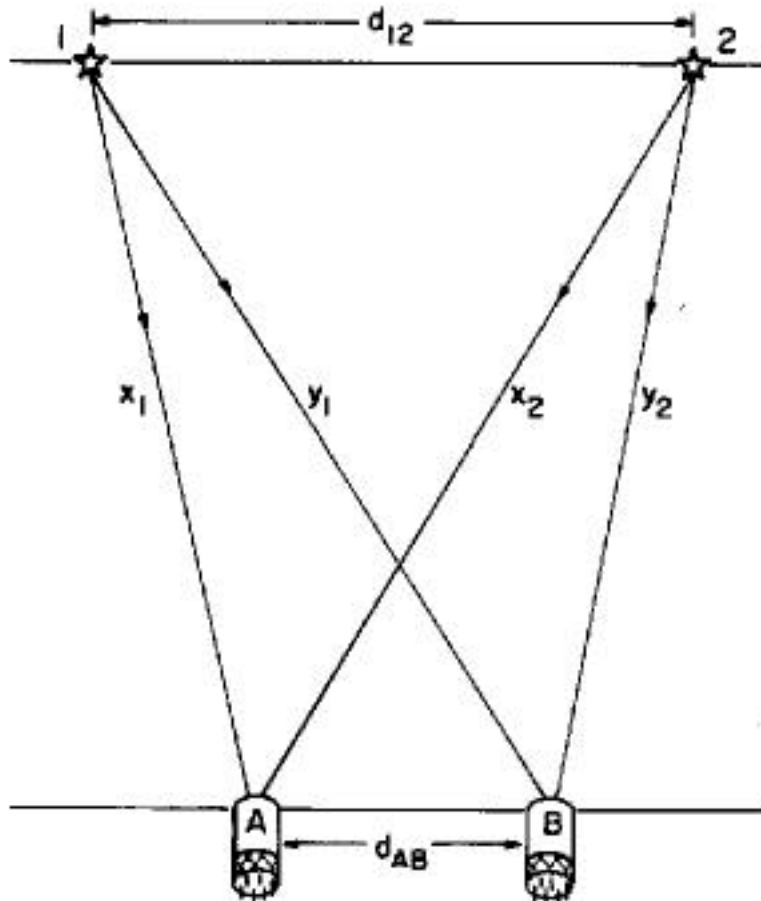
and, hence,

$$\langle \bar{I}^2 \rangle = 1 + \langle \bar{I} \rangle^2 \quad [\text{VIA-52}]$$

which is important result which holds for any chaotic source -- see the figure on next page. Notice that this result predicts a **bunching** of the intensity variations.

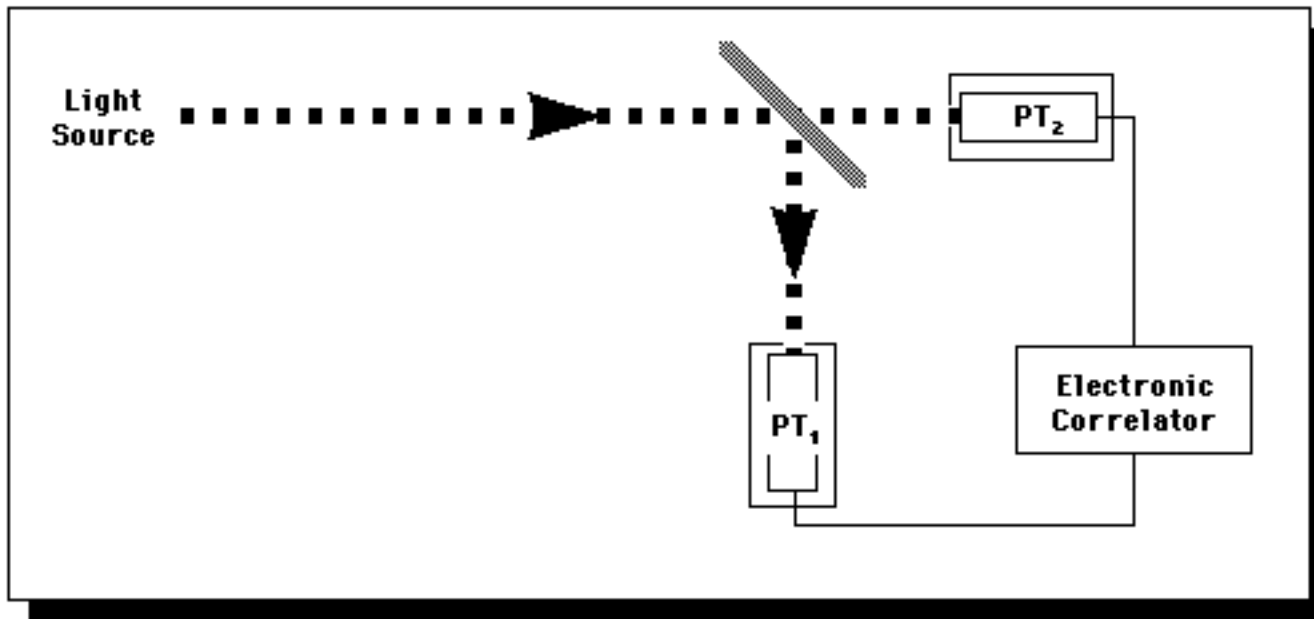


THE HANBURY BROWN - TWIST EXPERIMENT



Schematic diagram of the Hanbury Brown-Twiss experiment demonstrating coherent interference in light from incoherent sources. Sources 1 and 2 are separate by distance d_{12} , and emit photons of identical wavelengths which are detected by detectors A and B located a distance d_{AB} apart. The distance between the source plane and the detector plane is L . A coincidence between detector outputs results in a composite signal exhibiting an interference effect depending on both d_{12} and d_{AB} .

THE HANBURY BROWN - TWISS SPECTROMETER



The Hanbury Brown - Twiss Spectrometer is designed to measure¹²

$$\langle [\bar{I}_1(\vec{r}_{\text{ref}}, t_1) - \bar{I}_1] [\bar{I}_2(\vec{r}_{\text{ref}}, t_2) - \bar{I}_2] \rangle = \frac{1}{4} \left\{ \langle \bar{I}_1(\vec{r}_{\text{ref}}, t_1) \bar{I}_2(\vec{r}_{\text{ref}}, t_2) \rangle - \bar{I}^2 \right\} \quad [\text{VIA-53}]$$

This correlation may then be written

$$\frac{\langle [\bar{I}_1(\vec{r}_{\text{ref}}, t_1) - \bar{I}_1] [\bar{I}_2(\vec{r}_{\text{ref}}, t_2) - \bar{I}_2] \rangle}{\bar{I}_1 \bar{I}_2} = {}^{(2)}() - 1 \quad [\text{VIA-54}]$$

Generally, the results of these experiments are consistent with Equation [VIA-52], but some of the more detailed observations require a quantum mechanical interpretation.

¹² R. Hanbury Brown and R. Q. Twiss, *Proc. Roy. Soc., A* **242**, 300 (1957).

