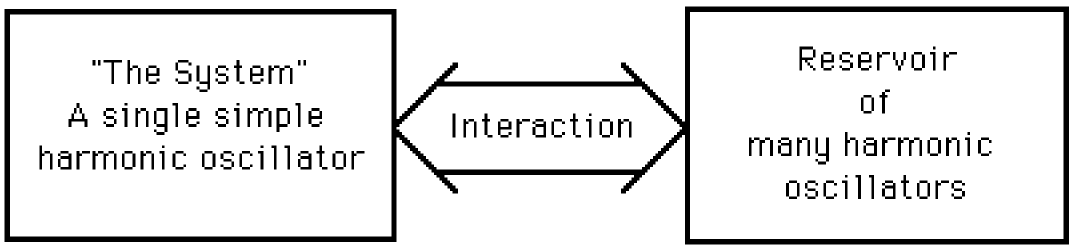
VII. QUANTUM MECHANICAL LANGEVIN EQUATIONS [38]

A RUDIMENTARY RESEVOIR PROBLEM:[39][40]

Consider a complete system which consists of a single simple harmonic oscillator (the system component of "interest") coupled to a ***reservoir*** of many simple harmonic oscillators (the system component treated statistically)



In particular, the complete interacting system has a simple Hamiltonian of the form

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{laser} + \mathcal{H}_{res} + \mathcal{H}_{\text{int}} \\ &= \hbar\Omega a^\dagger a + \sum_j \hbar\omega_j b_j^\dagger b_j + \sum_j \hbar\left\{g_j a^\dagger b_j + g_j b_j^\dagger a\right\} \end{aligned} \qquad [\text{ VII-1 }]$$

where a is the system variable of *interest* (the "state of the system") and the b_j's are the *other* system variables (the "reservoir states"). System operators commute with reservoir operators at given time, we may easily generate the following set of equations of motion:

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[38] Much of what follows draws heavily on material in the ***Arizona Books*** -- *i.e.*

1. M. Sargent III, M. O. Scully and W. E. Lamb, Jr., *Laser Physics*, Addison-Wesley (1974)
2. P. Meystre and M. Sargent III, *Elements of Quantum Optics*, Springer-Verlag (1992)
3. Weng W. Chow, Stephan W. Koch and Murray Sargent III, *Semiconductor-Laser Physics*, Springer-Verlag (1994)

[39] This model is particular valuable in treating electromagnetic intermode coupling problems, but it also provides guidence in treating more general problems which incorporate atomic states.

[40]

$$\dot{a}(t) = i\left[\mathcal{H}, a(t)\right]/\hbar = -i\,\Omega\, a(t) - i\sum_j g_j\, b_j(t) \qquad \text{[VII-2a]}$$

$$\dot{b}_j(t) = i\left[\mathcal{H}, b_j(t)\right]/\hbar = -i\,\omega_j\, b_j(t) - i g_j\, a(t) \qquad \text{[VII-2b]}$$

Formally integrating Equation [VII-2b], we obtain [41]

$$b_j(t) = b_j(t_0)\,\exp\left[-i\,\omega_j\,(t-t_0)\right] - i\, g_j \int_{t_0}^{t} dt'\, a(t')\exp\left[-i\,\omega_j\,(t-t')\right] \qquad \text{[VII-3]}$$

The first term in this equation describes the ***free*** evolution of the reservoir states in the absence of any interaction with the system and the second gives the modification of this free evolution as a consequence of the coupling to the system. Inserting Equation [VII-3] into Equation [VII-2a] we obtain

$$\begin{aligned}\dot{a}(t) = -i\,\Omega\, a(t) - \sum_j |\,g_j\,|^2 \int_{t_0}^{t} dt'\, a(t')\exp\left[-i\,\omega_j\,(t-t')\right] \\ - i\sum_j g_j\, b_j(t_0)\exp\left[-i\,\omega_j\,(t-t_0)\right]\end{aligned} \qquad \text{[VII-4]}$$

Here the second summation describes a source of fluctuations arising from the free evolution of the reservoir states and the first is a feedback through the reservoir which might be described as a *radiation reaction.* To remove the rapid variation in the system variable we transform to the Heisenberg interaction picture -- *i.e.* we take [42]

$$a(t) = A(t)\,\exp\left[-i\,\Omega\, t\right] \qquad \text{[VII-5]}$$

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41 As is pointed out in Section 14-3 of P. Meystre and M. Sargent III, *Elements of Quantum Optics*, Springer-Verlag (1992), the statement that "System operators commute with reservoir operators at given time…" is a very critical point. From Equation [II-3], we see that as time evolves, the reservoir operators acquire some of the characteristics of the system operator. Thus, it is crucial in this development to maintain a strict and consistent ordering of operators at different times!

42 The transformation does not alter the system's communtation relationship, so, of necessity, $\left[A(t), A^{\dagger}(t)\right] = 1$.

so that $$\dot{A}(t) = -\sum_j |\, g_j \,|^2 \int_{t_0}^{t} d\,t'\; A(t')\, \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right] + F(t) \qquad \text{[VII-6]}$$

where $$F(t) = -i \sum_j \; g_j \;\; b_j(t_0)\; \exp\left[\, i\left(\Omega - \omega_j\right)\left(t - t_0\right)\right] \qquad \text{[VII-7]}$$

is the so called ***noise operator***.

The first term on the RHS of Equation [VII-6] may be reordered as

$$\begin{aligned} &\sum_j |\, g_j \,|^2 \int_{t_0}^{t} d\,t'\; A(t')\, \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right] \\ &\quad = \int_{t_0}^{t} d\,t'\; A(t') \sum_j |\, g_j \,|^2 \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right] \end{aligned} \quad . \qquad \text{[VII-8a]}$$

By assumption, the frequencies of the reservoir oscillators are closely spaced and widely distribution so that the function $\sum_j |\, g_j \,|^2 \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right]$ is very sharply peaked at $t' = t$ and has a width of the order of the inverse of the reservoir's frequency bandwidth. Thus, $A(t')$ may taken out of the integration and the limits of the integration may be extended to infinite -- *i.e.*

$$\begin{aligned} &\sum_j |\, g_j \,|^2 \int_{t_0}^{t} d\,t'\; A(t')\, \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right] \\ &\quad A(t) \sum_j |\, g_j \,|^2 \int_{-\infty}^{+\infty} d\,t'\; \exp\left[-i\left(\omega_j - \Omega\right)\left(t - t'\right)\right] \end{aligned} \quad . \qquad \text{[VII-8b]}$$

Again following arguments explicated by Heitler,[43] we see that

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43 In Chapter II, Section 8 of W. Heitler, *The Quantum Theory of Radiation* (3rd edition)

$$\int_{-\infty}^{+\infty} d\,t' \; \exp\left[-i\left(\Omega-\omega_j\right)\left(t-t'\right)\right] \quad \pi \;\delta\left(\Omega-\omega_j\right) - \mathcal{P}r\; \frac{i}{\Omega-\omega_j} \qquad \text{[VII-9]}$$

If we neglect the imaginary part of Equation [VII-9], which leads to a Lamb-type frequency shift, Equation [VII-6] leads directly to a quantum mechanical version of the classical Langevin equation [44] -- *viz*.

$$\dot{A}\left(t\right) = -\frac{\gamma}{2} A\left(t\right) + F\left(t\right) \qquad \text{[VII-10]}$$

where

$$\gamma = 2\pi\,\rho\left(\Omega\right) \left|\, g\left(\Omega\right)\right|^2 . \qquad \text{[VII-11]}$$

is the quantum mechanical Langevin *drift coefficient*. In optical applications the average *intensity* or *number operator* -- *i.e.* $\left\langle A^{\dagger}\left(t\right) A\left(t\right)\right\rangle$ -- is the experimental quantity of greatest interest. A useful equation of motion for that operator may obtained by making use the Langevin equation -- *i.e.* Equation [VII-10]. To that end we first write

$$\frac{d}{d\,t}\left\langle A^{\dagger}\left(t\right) A\left(t\right)\right\rangle = -\gamma\left\langle A^{\dagger}\left(t\right) A\left(t\right)\right\rangle + \left\langle F^{\dagger}\left(t\right) A\left(t\right)\right\rangle + \left\langle A^{\dagger}\left(t\right) F\left(t\right)\right\rangle \qquad \text{[VII-12]}$$

To obtain $\left\langle F^{\dagger}\left(t\right) A\left(t\right)\right\rangle$ and $\left\langle A^{\dagger}\left(t\right) F\left(t\right)\right\rangle$ we make use of the identity

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44 In the classical Langevin theory of Browian motion, the equation of motion for drift velocity of a particle suspended in a liquid is given by

$$m\,\dot{u}\left(t\right) = -m\,\Gamma\,u\left(t\right) + F_u\left(t\right)$$

where Γ is a damping constant and $F_v\left(t\right)$ is a random, rapidly fluctuation force with zero mean value. In this classical problem the famous fluctuation-dissipation theorem is expressed as

$$\Gamma = \frac{\beta}{m} D_{uu} = \frac{\beta}{2\,m}\int_{-\infty}^{+\infty} d\tau \;\left\langle F_u\left(0\right) F_u\left(\tau\right)\right\rangle$$

where D_{uu} is the so called Fokker-Planck diffusion coefficient.

$$A(t) = A(t-\Delta t) + \int_{t-\Delta t}^{t} dt'\, \dot{A}(t') \qquad [\text{ VII-13 }]$$

so that

$$\begin{aligned}\langle F^\dagger(t)\, A(t)\rangle &= \langle F^\dagger(t)\, A(t-\Delta t)\rangle + \int_{t-\Delta t}^{t} dt' \langle F^\dagger(t)\, \dot{A}(t')\rangle \\ &= \langle F^\dagger(t)\, A(t-\Delta t)\rangle - \frac{\gamma}{2}\int_{t-\Delta t}^{t} dt' \langle F^\dagger(t)\, A(t')\rangle + \int_{t-\Delta t}^{t} dt' \langle F^\dagger(t)\, F(t')\rangle\end{aligned} \qquad [\text{ VII-14 }]$$

The first term on the RHS vanishes since *cause* cannot precede *effect*. That is, $A(t)$ cannot be correlated with a future value of the fluctuating force. Similarly, in the second term $\langle F^\dagger(t)\, A(t')\rangle$ vanishes except in the infinitesimal small inteval where $t' = t$. Therefore

$$\langle F^\dagger(t)\, A(t)\rangle = \int_{t-\Delta t}^{t} dt' \langle F^\dagger(t)\, F(t')\rangle\ . \qquad [\text{ VII-15a }]$$

If the random force is stationary in time

$$\langle F^\dagger(t)\, A(t)\rangle = \int_{-\Delta t}^{0} d\tau \langle F^\dagger(t)\, F(t+\tau)\rangle \qquad [\text{ VII-15b }]$$

and, finally, if Δt is long compared to the random force correlation time

$$\langle F^\dagger(t)\, A(t)\rangle = \int_{-\infty}^{0} d\tau\ \langle F^\dagger(t)\, F(t+\tau)\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \langle F^\dagger(0)\, F(\tau)\rangle\ . \qquad [\text{ VII-15c }]$$

Therefore Equation [VII-12], the equation of motion for the average number operator, becomes

$$\frac{d}{dt}\langle A^\dagger(t)\, A(t)\rangle = -\gamma \langle A^\dagger(t)\, A(t)\rangle + \int_{-\infty}^{+\infty} d\tau\ \langle F^\dagger(0), F(\tau)\rangle \qquad [\text{ VII-16 }]$$

which is one form of the famous fluctuation-dissipation theorem.

If the reservoir is in thermal equilibrium, we may directly evaluate the noise operator correlation to wit.

$$\left\langle F^{\dagger}(t)\, F(t')\right\rangle_{therm} = \sum_{j,\,k} g_j\ g_k \left\langle b_j^{\dagger}(0)\, b_k(0)\right\rangle_{therm} \exp\left[-i\left(\Omega-\omega_j\right)\, t\right] \exp\left[i\left(\Omega-\omega_k\right)\, t'\right] \qquad [\text{ VII-17 }]$$

Since the operators of the reservoir commute we can write

$$\begin{aligned} \left\langle b_j^{\dagger}(0)\, b_k(0)\right\rangle_{therm} &= \left\langle\, n_j\, \right\rangle_{therm} \delta_{jk} \\ &= \frac{\partial}{\partial\beta} \ln\left\{ \sum_{n_j} \exp\left[-n_j\, \beta\, \hbar\omega_j\right] \right\} \\ &= \left\{ \exp\left[\beta\, \hbar\, \omega_j\right] - 1 \right\}^{-1} \delta_{jk} \end{aligned} \qquad [\text{ VII-18 }]$$

and Equation [VII-17] becomes

$$\begin{aligned} \left\langle F^{\dagger}(t)\, F(t')\right\rangle_{therm} &= \sum_j |\, g_j\,|^2 \left\langle\, n_j\, \right\rangle_{therm} \exp\left[-i\left(\Omega-\omega_j\right)\left(t-t'\right)\right] \\ &= 2\,\pi\, |\, g(\Omega)\,|^2\, \rho\,(\Omega)\ \left\langle\, n(\Omega)\, \right\rangle_{therm} \delta\left(t-t'\right) \\ &= \gamma \left\langle\, n(\Omega)\, \right\rangle_{therm} \delta\left(t-t'\right) \\ &= 2\left\langle\, D_{A^{\dagger}A}\, \right\rangle_{therm} \delta\left(t-t'\right) \end{aligned} \qquad [\text{ VII-19 }]$$

where $\left\langle\, D_{A^{\dagger}A}\, \right\rangle_{therm} = \frac{\gamma}{4} \left\langle\, n\,(\Omega)\, \right\rangle_{therm}$ is of the form of a *diffusion coefficient* in the classical Langevin theory. Two-time correlation functions of this form are characteristic of **Markoffian** random processes and represent fluctuations in a reservoir with essentially zero memory. Substitution of this Markoffian correlation function into Equation [VII-16] yields an *Einstein relation* of the form

$$2\left\langle D_{A^\dagger A} \right\rangle_{therm} = \frac{d}{dt}\left\langle A^\dagger(t)\, A(t) \right\rangle_{therm} + \gamma \left\langle A^\dagger(t)\, A(t) \right\rangle_{therm} . \qquad \text{[VII-20]}$$

GENERAL RESEVOIR PROBLEM

As the previous section demonstrates, the effects of random fluctuations in an assemblage of harmonic oscillators may be accounted for in precise detail. The problem can be analyzed completely since the first term on the RHS of Equation [VII-6] -- the ***damping term*** -- contains only the system variable. In general, this feedback term will included reservoir variables as well and the simplified model breaks down. The general case may, however, be analyzed if we take the simpler case as a guide. In particular, let us suppose that a set of systems operators $\left\{ A_\mu(t) \right\}$ are coupled to the reservoir and satisfy a set Langevin equations of motion as

$$\dot{A}_\mu(t) = D_\mu(t) + F_\mu(t) \qquad \text{[VII-21]}$$

where $D_\mu(t)$ is the drift term and $F_\mu(t)$ is the Markoffian noise operator appropriate to the system operator $A_\mu(t)$. From the earlier results, we are, thus, assuming that the two-time correlation of the random fluctuating force is of the form

$$\left\langle F_\mu(t')\, F_\nu(t'') \right\rangle = 2\, D_{\mu\nu}\, \delta(t' - t'') \qquad \text{[VII-22]}$$

Again using the identity

$$A_\mu(t) = A_\mu(t - \Delta t) + \int_{t-\Delta t}^{t} dt'\, \dot{A}_\mu(t') \qquad \text{[VII-23]}$$

we find

$$\langle A_\mu(t)\,F_\nu(t)\rangle = \langle A_\mu(t-\Delta t)\,F_\nu(t)\rangle + \int_{t-\Delta t}^{t} dt'\,\langle D_\mu(t')\,F_\nu(t)\rangle + \int_{t-\Delta t}^{t} dt'\,\langle F_\mu(t')\,F_\nu(t)\rangle \qquad \text{[VII-24]}$$

As discussed in connection with Equation [VII-14], the first and second term on the RHS in Equation [VII-24] vanishes since *cause* cannot precede *effect*. Therefore

$$\langle A_\mu(t)\,F_\nu(t)\rangle = \int_{t-\Delta t}^{t} dt'\,\langle F_\mu(t')\,F_\nu(t)\rangle \qquad \text{[VII-25]}$$

and by Equation [VII-22]

$$\langle A_\mu(t)\,F_\nu(t)\rangle = D_{\mu\,\nu} \qquad \text{[VII-26]}$$

We may then form the following equation of motion

$$\frac{d}{dt}\langle A_\mu\,A_\nu\rangle = \langle \dot{A}_\mu\,A_\nu\rangle + \langle A_\mu\,\dot{A}_\nu\rangle = \langle D_\mu\,A_\nu\rangle + \langle F_\mu\,A_\nu\rangle + \langle A_\mu\,D_\nu\rangle + \langle A_\mu\,F_\nu\rangle \qquad \text{[VII-27]}$$

which leads to a *generalized Einstein relationship*

$$2\,D_{\mu\,\nu} = \frac{d}{dt}\langle A_\mu\,A_\nu\rangle - \langle D_\mu\,A_\nu\rangle - \langle A_\mu\,D_\nu\rangle \qquad \text{[VII-28]}$$

With this equation it is possible to calculate the diffusion coefficients from the drift coefficients, provided one can independently calculate the equation of motion for $\langle A_\mu\,A_\nu\rangle$.

We can also use Equation [VII-21] to calculate the expectation values of two-time correlation functions. In paryicular, if multiply it by $A_\nu(t')$ where $t' < t$ and average we find

$$\frac{d}{dt}\langle A_\mu(t)\,A_\nu(t')\rangle = \langle D_\mu(t)\,A_\nu(t')\rangle + \langle F_\mu(t)\,A_\nu(t')\rangle \qquad \text{[VII-29]}$$

Since in the Markoff approximation, the present system operator $A_\nu(t')$ cannot know about the future noise source $F_\mu(t)$, $\langle F_\mu(t)\,A_\nu(t')\rangle$ vanishes and

$$\frac{d}{dt}\langle A_\mu(t)\,A_\nu(t')\rangle = \langle D_\mu(t)\,A_\nu(t')\rangle \qquad \text{[VII-30]}$$

which is the so called quantum regression theorm which states that the two-time correlation function $\langle A_\mu(t)\,A_\nu(t')\rangle$ satisfies the same equation of motion as the single-time $\langle A_\mu(t)\rangle$.

SEMICONDUCTOR LANGEVIN EQUATIONS

As discussed earlier, the effective Hamiltonian for optical interactions in a semiconductor may be written as

$$\mathcal{H}_{\text{eff}} = \sum_{\vec{k}} \left\{ \left[\mathcal{E}_g + \frac{\hbar^2 k^2}{2m_{ec}} \right] a^{\dagger}_{\vec{k}} a_{\vec{k}} + \left[\frac{\hbar^2 k^2}{2m_{hv}} \right] b^{\dagger}_{-\vec{k}} b_{-\vec{k}} - \left[g_{\vec{k}} a^{\dagger}_{\vec{k}} b^{\dagger}_{-\vec{k}} a + g_{\vec{k}} a^{\dagger} a_{\vec{k}} b_{-\vec{k}} \right] \right\} \qquad \text{[VII-29]}$$

Thus, the equation of motion motion for the effective dipole operator $b_{-\vec{k}} a_{\vec{k}}$ is given by

$$\begin{aligned} \frac{d}{dt} b_{-\vec{k}} a_{\vec{k}} &= \frac{i}{\hbar}\left[\mathcal{H}_{\text{eff}}, b_{-\vec{k}} a_{\vec{k}} \right] \\ &= -\left(\gamma + i\,\omega_{\vec{k}}\right) b_{-\vec{k}} a_{\vec{k}} + i g_{\vec{k}}\, a \left(a^{\dagger}_{\vec{k}} a_{\vec{k}} + b^{\dagger}_{-\vec{k}} b_{-\vec{k}} - 1 \right) + f_{b_{-\vec{k}} a_{\vec{k}}} \end{aligned} \qquad \text{[VII-30]}$$

where γ is the damping, $\omega_{\vec{k}} = \mathcal{E}_g + \dfrac{\hbar^2 k^2}{2m_{ec}} + \dfrac{\hbar^2 k^2}{2m_{hv}}$, and $f_{b_{-\vec{k}} a_{\vec{k}}}$ is the k-dependent noise operator. We transform the various operators to a frame rotating at the electromagnetic field frequency ω_r -- *viz.*

$$b_{-\vec{k}} a_{\vec{k}} = \sigma_{\vec{k}} \exp\left[-\omega_r\, t\right] \qquad \text{[VII-31a]}$$

$$a(t) = A(t) \exp\left[-\omega_r\, t\right] \qquad \text{[VII-31b]}$$

$$f_{b_{-\vec{k}} a_{\vec{k}}} = f_{\sigma_{\vec{k}}} = F_{\sigma_{\vec{k}}} \exp\left[-\omega_r\, t\right] \qquad \text{[VII-31c]}$$

and find the transformed equation of motion in the rotating frame -- *i.e.* the Heisenberg interaction picture -- as

$$\frac{d}{dt}\sigma_{\vec{k}} = -\left(\gamma + i\,\omega_{\vec{k}} - i\,\omega_r\right) \sigma_{\vec{k}} + i g_{\vec{k}} A \left(a^{\dagger}_{\vec{k}} a_{\vec{k}} + b^{\dagger}_{-\vec{k}} b_{-\vec{k}} - 1 \right) + F_{\sigma_{\vec{k}}} \qquad \text{[VII-32]}$$

Applying the generalized Einstein relation -- *i.e.* Equation [VII-28] -- to the operator

$$\sigma^{\dagger}_{\mathbf{k}}\,\sigma_{\mathbf{k}} = a^{\dagger}_{\mathbf{k}}\, b^{\dagger}_{-\mathbf{k}}\, b_{-\mathbf{k}}\, a_{\mathbf{k}} = a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{k}}\, b^{\dagger}_{-\mathbf{k}}\, b_{-\mathbf{k}} \equiv \sigma^{\mathbf{k}}_{33}$$

we obtain

$$\begin{aligned} 2\,D_{\sigma^{\dagger}_{\mathbf{k}}\,\sigma_{\mathbf{k}}} &= \left(\gamma + i\,\omega_{\mathbf{k}} - i\,\omega_r\right)\sigma^{\mathbf{k}}_{33} + \left(\gamma - i\,\omega_{\mathbf{k}} + i\,\omega_r\right)\sigma^{\mathbf{k}}_{33} + \sigma^{\mathbf{k}}_{33} \\ &\approx 2\,\gamma\,\sigma^{\mathbf{k}}_{33} \end{aligned} \qquad \text{[VII-33a]}$$

Similarly,

$$\begin{aligned} 2\,D_{\sigma_{\mathbf{k}}\,\sigma^{\dagger}_{\mathbf{k}}} &= \left(\gamma + i\,\omega_{\mathbf{k}} - i\,\omega_r\right)\sigma^{\mathbf{k}}_{00} + \left(\gamma - i\,\omega_{\mathbf{k}} + i\,\omega_r\right)\sigma^{\mathbf{k}}_{00} + \sigma^{\mathbf{k}}_{00} \\ &\approx 2\,\gamma\,\sigma^{\mathbf{k}}_{00} \end{aligned} \qquad \text{[VII-33b]}$$

where $\sigma^{\mathbf{k}}_{00} = \left(1 - a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{k}}\right)\left(1 - b^{\dagger}_{-\mathbf{k}}\, b_{-\mathbf{k}}\right)$.

The equation of motion for the laser field operator in a frame rotating at the laser frequency ω_r may be written as

$$\dot{A}(t) = -\left[\frac{\omega}{2\,Q} + i\left(\Omega - \omega_r\right)\right]A(t) - i\sum_{\mathbf{k}} g_{\mathbf{k}}\,\sigma_{\mathbf{k}}(t) + F(t) \qquad \text{[VII-34]}$$

where Ω is the cold cavity resonant frequency and where we have replace the damping parameter γ in equation [VII-10] by the experimental parameter ω/Q . If we consider the Equation [VII-32] in the *quasi-equilbrium* limit, we find

$$\begin{aligned} \sigma_{\mathbf{k}}(t) &= \left[\gamma + i\,\omega_{\mathbf{k}} - i\,\omega_r\right]^{-1}\left[i g_{\mathbf{k}}\, A(t)\left(a^{\dagger}_{\mathbf{k}}\, a_{\mathbf{k}} + b^{\dagger}_{-\mathbf{k}}\, b_{-\mathbf{k}} - 1\right) + F_{\sigma_{\mathbf{k}}}(t)\right] \\ &= \mathcal{D}\left(\omega_{\mathbf{k}} - \omega_r;\gamma\right)\left[i g_{\mathbf{k}}\, A(t)\left(\sigma^{\mathbf{k}}_{33} - \sigma^{\mathbf{k}}_{00}\right) + F_{\sigma_{\mathbf{k}}}(t)\right] \end{aligned} \qquad \text{[VII-35]}$$

where $\mathcal{D}\left(\omega_{\vec{\mathbf{k}}}-\omega_r;\gamma\right)$ is the complex Lorentzian demoninator.[45] Thus in the quasi-equilbrium limit, Equation [VII-34] becomes

$$\dot{A}(t)=-\left[\frac{\omega}{2Q}+i\left(\Omega-\omega_r\right)-\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2\mathcal{D}\left(\omega_{\vec{\mathbf{k}}}-\omega_r;\gamma\right)\left(\sigma_{33}^{\vec{\mathbf{k}}}-\sigma_{00}^{\vec{\mathbf{k}}}\right)\right]A(t)+F_A(t) \qquad \text{[VII-36]}$$

where
$$F_A(t)=F(t)-i\sum_{\vec{\mathbf{k}}}g_{\vec{\mathbf{k}}}\,\mathcal{D}\left(\omega_{\vec{\mathbf{k}}}-\omega_r;\gamma\right)F_{\sigma_{\vec{\mathbf{k}}}}(t)\ . \qquad \text{[VII-37]}$$

Following arguments similar to those used in deriving Equation [VII-16], we easily derive an equation of motion for the laser photon number operator -- *viz*.

$$\frac{d}{dt}\left\langle A^\dagger(t)\,A(t)\right\rangle=-\left[\frac{\omega}{Q}-\frac{2}{\gamma}\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2\mathcal{L}\left(\omega_{\vec{\mathbf{k}}}-\omega_r;\gamma\right)\left(\sigma_{33}-\sigma_{00}\right)\right]\left\langle A^\dagger(t)\,A(t)\right\rangle+\int_{-\infty}^{+\infty}d\tau\left\langle F_A^\dagger(0),F_A(\tau)\right\rangle \qquad \text{[VII-38a]}$$

or

$$\frac{d}{dt}\left\langle A^\dagger(t)\,A(t)\right\rangle=-\left[\frac{\omega}{Q}-\left(R_{33}-R_{00}\right)\right]\left\langle A^\dagger(t)\,A(t)\right\rangle+\int_{-\infty}^{+\infty}d\tau\left\langle F_A^\dagger(0),F_A(\tau)\right\rangle \qquad \text{[VII-38b]}$$

where
$$R_{33}=\frac{2}{\gamma}\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2\mathcal{L}\left(\omega_{\vec{\mathbf{k}}}-\omega_r;\gamma\right)\sigma_{33}^{\vec{\mathbf{k}}}\ . \qquad \text{[VII-39a]}$$

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45 It may be recalled that in the notes entitled *Lasers: Models and Theories* for convenience we indroduced the Lorentzian functions -- *viz*.

$$\mathcal{D}(u-v;w)\equiv\left[i\left(u-v\right)+w\right]^{-1} \qquad \text{the complex Lorentzian denominator}$$

$$\mathcal{L}(u-v;w)\equiv w^2\left[\left(u-v\right)^2+w^2\right]^{-1} \qquad \text{the dimensionless Lorentzian function.}$$

and
$$R_{00} = \frac{2}{\gamma}\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2 \mathcal{L}\left(\omega_{\vec{\mathbf{k}}} - \omega_r ; \gamma\right) \sigma_{00}^{\vec{\mathbf{k}}} \ . \qquad \text{[VII-39b]}$$

Assuming that the cavity damping and carrier scattering reservoirs are uncorrelated, the two-time correlation function may be written as

$$\begin{aligned}\left\langle F_A^\dagger(t)\ F_A(t')\right\rangle &= \left\langle F^\dagger(t)\ F(t')\right\rangle \\ &+ \sum_{\vec{\mathbf{k}}}\sum_{\vec{\mathbf{k}}'} g_{\vec{\mathbf{k}}} g_{\vec{\mathbf{k}}'}\ \mathcal{D}\left(\omega_{\vec{\mathbf{k}}'} - \omega_r ; \gamma\right) \mathcal{D}\left(\omega_{\vec{\mathbf{k}}} - \omega_r ; \gamma\right) \left\langle F_{\sigma_{\vec{\mathbf{k}}}}^\dagger(t)\ F_{\sigma_{\vec{\mathbf{k}}}}(t')\right\rangle \end{aligned} \qquad \text{[VII-40a]}$$

and if the noise operators for different $\vec{\mathbf{k}}$ are uncorrelated

$$\left\langle F_A^\dagger(t)\ F_A(t')\right\rangle = \left\langle F^\dagger(t)\ F(t')\right\rangle + \frac{1}{\gamma^2}\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2 \mathcal{L}\left(\omega_{\vec{\mathbf{k}}} - \omega_r ; \gamma\right) \left\langle F_{\sigma_{\vec{\mathbf{k}}}}^\dagger(t)\ F_{\sigma_{\vec{\mathbf{k}}'}}(t')\right\rangle \ . \qquad \text{[VII-40b]}$$

Finally, using Equations [VII-19] and [VII-33a] we obtain

$$\left\langle F_A^\dagger(t)\ F_A(t')\right\rangle = \left[\frac{\omega}{Q}\left\langle n(\omega)\right\rangle_{therm} + \frac{2}{\gamma}\sum_{\vec{\mathbf{k}}}\left|g_{\vec{\mathbf{k}}}\right|^2 \mathcal{L}\left(\omega_{\vec{\mathbf{k}}} - \omega_r ; \gamma\right) \sigma_{33}^{\vec{\mathbf{k}}}\right] \delta(t - t') \ . \qquad \text{[VII-40c]}$$

and, thus, the "$A^\dagger A$" diffusion coefficient for the laser field is

$$2\,D_{A^\dagger A} = \frac{\omega}{Q}\left\langle n(\omega)\right\rangle_{therm} + R_{33} \ . \qquad \text{[VII-41a]}$$

By a similar argument, the "$A A^\dagger$" diffusion coefficient is

$$2\,D_{A A^\dagger} = \frac{\omega}{Q}\left[\left\langle n(\omega)\right\rangle_{therm} + 1\right] + R_{00} \ . \qquad \text{[VII-41b]}$$

Using Equation [VII-41a], Equation [VII-38b] becomes

$$\frac{d}{dt}\left\langle A^{\dagger}(t)\,A(t)\right\rangle = -\left[\frac{\omega}{Q} - \left(R_{33} - R_{00}\right)\right]\left\langle A^{\dagger}(t)\,A(t)\right\rangle + \frac{\omega}{Q}\left\langle n(\omega)\right\rangle_{therm} + R_{33} \qquad \text{[VII-42a]}$$

Similarly

$$\frac{d}{dt}\left\langle A(t)\,A^{\dagger}(t)\right\rangle = -\left[\frac{\omega}{Q} - \left(R_{33} - R_{00}\right)\right]\left\langle A(t)\,A^{\dagger}(t)\right\rangle + \frac{\omega}{Q}\left[\left\langle n(\omega)\right\rangle_{therm} + 1\right] + R_{00} \qquad \text{[VII-42b]}$$

As check on the consistency of these result, we see that the form of Equations [VII-42a] and [VII-42b] guarantees that the commutator $\left[A(t),\, A^{\dagger}(t)\right]$ is independent of time -- *viz*.

$$\frac{d}{dt}\left\langle\left[A(t), A^{\dagger}(t)\right]\right\rangle = -\left[\frac{\omega}{Q} - \left(R_{33} - R_{00}\right)\right]\left\langle\left[A(t), A^{\dagger}(t)\right]\right\rangle + \frac{\omega}{Q} + \left(R_{00} - R_{33}\right) = 0 \qquad \text{[VII-43]}$$

to incorporate saturation effects into the noise spectrum we need to write a Langevin equation for the carrier-density operator -- *viz*.

$$\begin{aligned} \frac{d}{dt}\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}} &= \frac{i}{\hbar}\left[\mathcal{H}_{\text{eff}},\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\right] + \left\{\text{damping, pumping, and noise terms}\right\} \\ &= \Lambda_e^{\vec{\mathbf{k}}}\left[1 - a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\right] - \Gamma_{\vec{\mathbf{k}}}\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\, b^{\dagger}_{-\vec{\mathbf{k}}}\, b_{-\vec{\mathbf{k}}} - \gamma_{nr}\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}} \\ &\quad - \gamma_e\left[a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}} - \left(a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\right)_{eq}\right] - i\left[g_{\vec{\mathbf{k}}}\, a^{\dagger}_{\vec{\mathbf{k}}}\, b^{\dagger}_{-\vec{\mathbf{k}}}\, a - g_{\vec{\mathbf{k}}}\, a^{\dagger}\, a_{\vec{\mathbf{k}}}\, b_{-\vec{\mathbf{k}}}\right] + F_e^{\vec{\mathbf{k}}} \end{aligned} \qquad \text{[VII-44]}$$

where $\Lambda_e^{\vec{\mathbf{k}}}\left[1 - a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\right]$ is the pumping due injected carriers, $\Gamma_{\vec{\mathbf{k}}}\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\, b^{\dagger}_{-\vec{\mathbf{k}}}\, b_{-\vec{\mathbf{k}}}$ is the radiative recombination (spontaneous emission) term, $\gamma_{nr}\, a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}$ is the nonradiative recombination term, $\gamma_e\left[a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\, b^{\dagger}_{-\vec{\mathbf{k}}} - \left(a^{\dagger}_{\vec{\mathbf{k}}}\, a_{\vec{\mathbf{k}}}\right)_{eq}\right]$ is the relaxation due carrier-carrier scattering, and $F_e^{\vec{\mathbf{k}}}$ is the carrier noise operator. In the quasi-equilibrium approximation -- *i.e.* from Equation [VII-35] -- we may sum Equation III-44] to obtain a Langevin equation for the total carrier-density operator

$$\frac{d}{dt} N = \Lambda - \frac{1}{V} \sum_{\vec{\mathbf{k}}} \Gamma_{\vec{\mathbf{k}}} \, a^{\dagger}_{\vec{\mathbf{k}}} \, a_{\vec{\mathbf{k}}} \, b^{\dagger}_{-\vec{\mathbf{k}}} \, b_{-\vec{\mathbf{k}}} - \gamma_{nr} \, N$$
$$- \frac{2 \, A^{\dagger} A}{\gamma \, V} \sum_{\vec{\mathbf{k}}} \left| g_{\vec{\mathbf{k}}} \right|^2 \mathcal{L}\left(\omega_{\vec{\mathbf{k}}} - \omega_r \, ; \gamma\right) \left(\sigma^{\vec{\mathbf{k}}}_{33} - \sigma^{\vec{\mathbf{k}}}_{00}\right) + F_N \qquad \text{[VII-45a]}$$

where

$$F_N(t) = F_e^{\vec{\mathbf{k}}}(t) + \frac{i}{V} \sum_{\vec{\mathbf{k}}} \left[g_{\vec{\mathbf{k}}} \, A^{\dagger}(t) \, \mathcal{D}\left(\omega_{\vec{\mathbf{k}}} - \omega_r \, ; \gamma\right) F_{\sigma_{\vec{\mathbf{k}}}}(t) - g_{\vec{\mathbf{k}}} \, A(t) \, \mathcal{D}\left(\omega_{\vec{\mathbf{k}}} - \omega_r \, ; \gamma\right) F^{\dagger}_{\sigma_{\vec{\mathbf{k}}}}(t) \right] \qquad \text{[VII-45b]}$$

NOISE SPECTRA:

To calculate noise spectra we make use of semiclassical transformations -- *viz.*

$$A(t) = \left(E(t) / \mathcal{E}_{\omega_r}\right) \exp\left[-i \, \varphi(t)\right] \qquad \text{[VII-46a]}$$

$$A^{\dagger}(t) = \left(E(t) / \mathcal{E}_{\omega_r}\right) \exp\left[\, i \, \varphi(t)\right] \qquad \text{[VII-46b]}$$

$$E(t) = \mathcal{E}_{\omega_r} \sqrt{A^{\dagger}(t) \, A(t)} \qquad \text{[VII-46c]}$$

$$\varphi(t) = (i/2) \, \ln\left[A(t) / A^{\dagger}(t)\right] \qquad \text{[VII-46d]}$$

where $\mathcal{E}_{\omega_r} = \left[\hbar \, \omega_r / \varepsilon_0 \, V\right]$ is the electric field per photon. Using the expression

$$D'_{\rho\sigma} = \sum_{\mu, \nu} D_{\mu\nu} \frac{\partial A'_{\rho}}{\partial A'_{\mu}} \frac{\partial A'_{\sigma}}{\partial A'_{\nu}} \qquad \text{[VII-47]}$$

we see that

$$D_{EE} = D_{A^\dagger A} \frac{\partial E}{\partial A^\dagger}\frac{\partial E}{\partial A} + D_{AA^\dagger}\frac{\partial E}{\partial A}\frac{\partial E}{\partial A^\dagger}$$
$$= \frac{\mathcal{E}_{\omega_r}^4}{4\,E^2}\left[AA^\dagger D_{A^\dagger A} + A^\dagger A\, D_{AA^\dagger}\right] \approx \frac{\mathcal{E}_{\omega_r}^4}{4}\left[D_{A^\dagger A} + D_{AA^\dagger}\right] \qquad \text{[VII-48a]}$$

$$D_{\varphi\varphi} = D_{A^\dagger A} \frac{\partial\varphi}{\partial A^\dagger}\frac{\partial\varphi}{\partial A} + D_{AA^\dagger}\frac{\partial\varphi}{\partial A}\frac{\partial\varphi}{\partial A^\dagger}$$
$$= \frac{1}{4\,AA^\dagger} D_{A^\dagger A} + \frac{1}{4\,A^\dagger A} D_{AA^\dagger} \approx \frac{\mathcal{E}_{\omega_r}^2}{4\,E^2}\left[D_{A^\dagger A} + D_{AA^\dagger}\right] \qquad \text{[VII-48a]}$$