VII. QUANTUM MECHANICAL LANGEVIN EQUATIONS

A RUDIMENTARY RESERVOIR PROBLEM: Consider a complete system which consists of a single simple harmonic oscillator (the system component of "interest") coupled to a reservoir of many simple harmonic oscillators (the system component treated statistically)

\[
\mathcal{H} = \mathcal{H}_{\text{laser}} + \mathcal{H}_{\text{res}} + \mathcal{H}_{\text{int}} = \hbar \omega a^\dagger a + \sum_j \hbar \omega_j b_j^\dagger b_j + \sum_j \hbar \left\{ g_j a^\dagger b_j + g_j^* b_j^\dagger a \right\} \tag{VII-1}
\]

where \(a\) is the system variable of interest (the "state of the system") and the \(b_j\)'s are the other system variables (the "reservoir states"). System operators commute with reservoir operators at given time, we may easily generate the following set of equations of motion:

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38 Much of what follows draws heavily on material in the Arizona Books -- i.e.

39 This model is particular valuable in treating electromagnetic intermode coupling problems, but it also provides guidance in treating more general problems which incorporate atomic states.

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\[ \dot{a}(t) = i \left[ \mathcal{H}, a(t) \right] / \hbar = -i \Omega a(t) - i \sum_j g_j b_j(t) \quad [\text{VII-2a}] \]

\[ \dot{b}_j(t) = i \left[ \mathcal{H}, b_j(t) \right] / \hbar = -i \omega_j b_j(t) - i g_j^* a(t) \quad [\text{VII-2b}] \]

Formally integrating Equation \[\text{VII-2b}\], we obtain \[41\]

\[ b_j(t) = b_j(t_0) \exp \left[ -i \omega_j (t - t_0) \right] - i \sum_j g_j^* \int_{t_0}^t dt' a(t') \exp \left[ -i \omega_j (t - t') \right] \quad [\text{VII-3}] \]

The first term in this equation describes the free evolution of the reservoir states in the absence of any interaction with the system and the second gives the modification of this free evolution as a consequence of the coupling to the system. Inserting Equation \[\text{VII-3}\] into Equation \[\text{VII-2a}\] we obtain

\[ \dot{a}(t) = -i \Omega a(t) - \sum_j g_j \int_{t_0}^t dt' a(t') \exp \left[ -i \omega_j (t - t') \right] \]

\[ - \sum_j g_j b_j(t_0) \exp \left[ -i \omega_j (t - t_0) \right] \quad [\text{VII-4}] \]

Here the second summation describes a source of fluctuations arising from the free evolution of the reservoir states and the first is a feedback through the reservoir which might be described as a radiation reaction. To remove the rapid variation in the system variable we transform to the Heisenberg interaction picture -- i.e. we take \[42\]

\[ a(t) = A(t) \exp \left[ -i \Omega \hat{a} \right] \quad [\text{VII-5}] \]

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\[41\] As is pointed out in Section 14-3 of P. Meystre and M. Sargent III, *Elements of Quantum Optics*, Springer-Verlag (1992), the statement that "System operators commute with reservoir operators at given time..." is a very critical point. From Equation \[\text{II-3}\], we see that as time evolves, the reservoir operators acquire some of the characteristics of the system operator. Thus, it is crucial in this development to maintain a strict and consistent ordering of operators at different times!

\[42\] The transformation does not alter the system's commutation relationship, so, of necessity, \[\left[ A(t), A(t') \right] = 1 \].
so that  
\[
\dot{A}(t) = -\sum_j g_j^2 \int_{t_0}^t dt\ A(t') \exp[-i (\omega_j - \Omega)(t - t')] + F(t) \tag{VII-6}
\]
where  
\[
F(t) = -i \sum_j g_j b_j(t_0) \exp[i (\Omega - \omega_j)(t - t_0)] \tag{VII-7}
\]
is the so called *noise operator*.

The first term on the RHS of Equation [ VII-6 ] may be reordered as

\[
\sum_j |g_j|^2 \int_{t_0}^t dt' A(t') \exp[-i (\omega_j - \Omega)(t - t')] = \int_{t_0}^t dt' A(t') \sum_j |g_j|^2 \exp[-i (\omega_j - \Omega)(t - t')] \tag{VII-8a}
\]

By assumption, the frequencies of the reservoir oscillators are closely spaced and widely distributed so that the function  
\[
\sum_j |g_j|^2 \exp[-i (\omega_j - \Omega)(t - t')] \]
is very sharply peaked at \( t' = t \) and has a width of the order of the inverse of the reservoir's frequency bandwidth. Thus,  \( A(t') \) may taken out of the integration and the limits of the integration may be extended to infinite -- i.e.

\[
\sum_j |g_j|^2 \int_{t_0}^t dt' A(t') \exp[-i (\omega_j - \Omega)(t - t')] = A(t) \sum_j |g_j|^2 \int_{-\infty}^{+\infty} dt' \exp[-i (\omega_j - \Omega)(t - t')] \tag{VII-8b}
\]

Again following arguments explicated by Heitler,\(^{43}\) we see that

\(^{43}\) In Chapter II, Section 8 of W. Heitler, *The Quantum Theory of Radiation* (3rd edition)
\[
\int_{-\infty}^{\infty} dt' \exp\left[-i(\Omega - \omega_j)(t - t')\right] \Rightarrow \pi \delta(\Omega - \omega_j) - \frac{i}{\Omega - \omega_j} \quad \text{[VII-9]}
\]

If we neglect the imaginary part of Equation [VII-9], which leads to a Lamb-type frequency shift, Equation [VII-6] leads directly to a quantum mechanical version of the classical Langevin equation\(^{44}\) — *viz.*

\[
\dot{A}(t) = -\frac{\gamma}{2} A(t) + F(t) \quad \text{[VII-10]}
\]

where

\[
\gamma = 2\pi \rho(\Omega) \left| g(\Omega) \right|^2. \quad \text{[VII-11]}
\]

is the quantum mechanical Langevin drift coefficient. In optical applications the average intensity or number operator — *i.e.* \(\langle \hat{A}^\dagger t A(t) \rangle\) — is the experimental quantity of greatest interest. A useful equation of motion for that operator may obtained by making use the Langevin equation — *i.e.* Equation [VII-10]. To that end we first write

\[
\frac{d}{dt} \langle \hat{A}^\dagger t A(t) \rangle = -\gamma \langle \hat{A}^\dagger t A(t) \rangle + \langle F^\dagger(t) A(t) \rangle + \langle \hat{A}^\dagger(t) F(t) \rangle \quad \text{[VII-12]}
\]

To obtain \(\langle F^\dagger(t) A(t) \rangle\) and \(\langle \hat{A}^\dagger(t) F(t) \rangle\) we make use of the identity

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\(^{44}\) In the classical Langevin theory of Brownian motion, the equation of motion for drift velocity of a particle suspended in a liquid is given by

\[
m \dot{u}(t) = -m \Gamma u(t) + F_u(t)
\]

where \(\Gamma\) is a damping constant and \(F_u(t)\) is a random, rapidly fluctuation force with zero mean value. In this classical problem the famous fluctuation-dissipation theorem is expressed as

\[
\Gamma = \frac{\beta}{m} D_{uu} = \frac{\beta}{2m} \int_{-\infty}^{\infty} d\tau \langle F_u(0) F_u(\tau) \rangle
\]

where \(D_{uu}\) is the so called Fokker-Planck diffusion coefficient.
\[ A(t) = A(t - \Delta t) + \int_{t-\Delta t}^{t} d\tau' \dot{A}(\tau') \]  

[ VII-13 ]

so that

\[
\left\{ F(t) A(t) \right\} = \left\{ F(t) A(t - \Delta t) \right\} + \int_{t-\Delta t}^{t} d\tau' \left\{ F(t) \dot{A}(\tau') \right\} 
\]

\[
= \left\{ F(t) A(t - \Delta t) \right\} - \gamma \int_{t-\Delta t}^{t} d\tau' \left\{ F(t) A(\tau') \right\} + \int_{t-\Delta t}^{t} d\tau' \left\{ F(t) F(\tau') \right\} 
\]

[ VII-14 ]

The first term on the RHS vanishes since *cause* cannot precede *effect*. That is, \( A(t) \) cannot be correlated with a future value of the fluctuating force. Similarly, in the second term \( \left\{ F(t) A(\tau') \right\} \) vanishes except in the infinitesimal small interval where \( \tau' = t \).

Therefore

\[
\left\{ F(t) A(t) \right\} = \int_{t-\Delta t}^{t} d\tau' \left\{ F(t) F(\tau') \right\}. 
\]

[ VII-15a ]

If the random force is stationary in time

\[
\left\{ F(t) A(t) \right\} = \int_{-\Delta t}^{0} d\tau \left\{ F(t) F(t + \tau) \right\} 
\]

[ VII-15b ]

and, finally, if \( \Delta t \) is long compared to the random force correlation time

\[
\left\{ F(t) A(t) \right\} = \int_{-\infty}^{0} d\tau \left\{ F(t) F(t + \tau) \right\} = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \left\{ F(0) F(\tau) \right\}. 
\]

[ VII-15c ]

Therefore Equation [ VII-12 ], the equation of motion for the average number operator, becomes

\[
\frac{d}{dt} \left\{ A(t) A(t) \right\} = -\gamma \left\{ A(t) A(t) \right\} + \int_{-\infty}^{+\infty} d\tau \left\{ F(0), F(\tau) \right\} 
\]

[ VII-16 ]
which is one form of the famous fluctuation-dissipation theorem.

If the reservoir is in thermal equilibrium, we may directly evaluate the noise operator correlation to wit.

\[
\{ F^\dagger(t) F(t) \}_{\text{therm}} = \sum_{j,k} g_j^* g_k \left\{ b_j(0) b_k(0) \right\}_{\text{therm}} \exp \left[ -i(\Omega - \omega_j) t \right] \exp \left[ i(\Omega - \omega_k) t' \right] \quad [\text{VII-17}]
\]

Since the operators of the reservoir commute we can write

\[
\left\{ b_j(0) b_k(0) \right\}_{\text{therm}} = \left\{ n_j \right\}_{\text{therm}} \delta_{jk}
\]

\[
= \frac{\partial}{\partial \beta} \ln \left\{ \sum_{n_j} \exp \left[ -n_j \beta \hbar \omega_j \right] \right\} \quad [\text{VII-18}]
\]

\[
= \left\{ \exp[\beta \hbar \omega_j] - 1 \right\}^{-1} \delta_{jk}
\]

and Equation [VII-17] becomes

\[
\{ F^\dagger(t) F(t) \}_{\text{therm}} = \sum_j g_j \left\{ n_j \right\}_{\text{therm}} \exp \left[ -i(\Omega - \omega_j) (t - t') \right]
\]

\[
= 2 \pi \left| g(\Omega) \right|^2 \rho(\Omega) \left\{ n(\Omega) \right\}_{\text{therm}} \delta(t - t')
\]

\[
= \gamma \left\{ n(\Omega) \right\}_{\text{therm}} \delta(t - t')
\]

\[
= 2 \left\{ D_{\text{A} A} \right\}_{\text{therm}} \delta(t - t')
\]

where \( \left\{ D_{\text{A} A} \right\}_{\text{therm}} = \frac{\gamma}{4} \left\{ n(\Omega) \right\}_{\text{therm}} \) is of the form of a diffusion coefficient in the classical Langevin theory. Two-time correlation functions of this form are characteristic of Markovian random processes and represent fluctuations in a reservoir with essentially
zero memory. Substitution of this Markoffian correlation function into Equation [ VII-16 ] yields an Einstein relation of the form

\[ 2 \left\{ D_{\alpha' \lambda} A_{\lambda} \right\}_\text{therm} = \frac{d}{dt} \left\{ A_{\lambda} A(t) \right\}_\text{therm} + \gamma \left\{ A_{\lambda} A(t) \right\}_\text{therm}. \]  

[ VII-20 ]

**General Reservoir Problem**

As the previous section demonstrates, the effects of random fluctuations in an assemblage of harmonic oscillators may be accounted for in precise detail. The problem can be analyzed completely since the first term on the RHS of Equation [ VII-6 ] -- the damping term -- contains only the system variable. In general, this feedback term will included reservoir variables as well and the simplified model breaks down. The general case may, however, be analyzed if we take the simpler case as a guide. In particular, let us suppose that a set of systems operators \( \{ A_\mu(t) \} \) are coupled to the reservoir and satisfy a set Langevin equations of motion as

\[ \dot{A}_\mu(t) = D_\mu(t) + F_\mu(t) \]  

[ VII-21 ]

where \( D_\mu(t) \) is the drift term and \( F_\mu(t) \) is the Markoffian noise operator appropriate to the system operator \( A_\mu(t) \). From the earlier results, we are, thus, assuming that the two-time correlation of the random fluctuating force is of the form

\[ \left\{ F_\mu(t') F_\nu(t'') \right\} = 2 D_{\mu \nu} \delta(t' - t'') \]  

[ VII-22 ]

Again using the identity

\[ A_\mu(t) = A_\mu(t - \Delta t) + \int_{t-\Delta t}^{t} d t' \dot{A}_\mu(t') \]  

[ VII-23 ]

we find
\[
\langle A_\mu (t) F_\nu (t) \rangle = \langle A_\mu (t-\Delta t) F_\nu (t) \rangle 
+ \int_{t-\Delta t}^t dt' \langle D_\mu (t') F_\nu (t) \rangle + \int_{t-\Delta t}^t dt' \langle F_\mu (t') F_\nu (t) \rangle
\]  

[ VII-24 ]

As discussed in connection with Equation [ VII-14 ], the first and second term on the RHS in Equation [ VII-24 ] vanishes since cause cannot precede effect. Therefore

\[
\langle A_\mu (t) F_\nu (t) \rangle = \int_{t-\Delta t}^t dt' \langle F_\mu (t') F_\nu (t) \rangle
\]  

[ VII-25 ]

and by Equation [ VII-22 ]

\[
\langle A_\mu (t) F_\nu (t) \rangle = D_{\mu \nu}
\]  

[ VII-26 ]

We may then form the following equation of motion

\[
\frac{d}{dt} \langle A_\mu A_\nu \rangle = \langle \dot{A}_\mu A_\nu \rangle + \langle A_\mu \dot{A}_\nu \rangle 
= \langle D_\mu A_\nu \rangle + \langle F_\mu A_\nu \rangle + \langle A_\mu D_\nu \rangle + \langle A_\mu F_\nu \rangle
\]  

[ VII-27 ]

which leads to a generalized Einstein relationship

\[
2 D_{\mu \nu} = \frac{d}{dt} \langle A_\mu A_\nu \rangle - \langle D_\mu A_\nu \rangle - \langle A_\mu D_\nu \rangle
\]  

[ VII-28 ]

With this equation it is possible to calculate the diffusion coefficients from the drift coefficients, provided one can independently calculate the equation of motion for \( \langle A_\mu A_\nu \rangle \).

We can also use Equation [ VII-21 ] to calculate the expectation values of two-time correlation functions. In particular, if multiply it by \( A_\nu (t') \) where \( t' < t \) and average we find
\[
\frac{d}{dt} \langle A_\mu (t) A_\nu (t') \rangle = \langle D_\mu (i) A_\nu (i') \rangle + \langle F_\mu (i) A_\nu (i') \rangle \quad \text{[ VII-29 ]}
\]

Since in the Markoff approximation, the present system operator \( A_\nu (i') \) cannot know about the future noise source \( F_\mu (i) \), \( \langle F_\mu (i) A_\nu (i') \rangle \) vanishes and

\[
\frac{d}{dt} \langle A_\mu (t) A_\nu (t') \rangle = \langle D_\mu (i) A_\nu (i') \rangle \quad \text{[ VII-30 ]}
\]

which is the so called quantum regression theorem which states that the two-time correlation function \( \langle A_\mu (i) A_\nu (i') \rangle \) satisfies the same equation of motion as the single-time \( \langle A_\mu (i) \rangle \).
SEMICONDUCTOR LANGEVIN EQUATIONS

As discussed earlier, the effective Hamiltonian for optical interactions in a semiconductor may be written as

$$\mathcal{H}_{\text{eff}} = \sum_k \left[ \left( \mathcal{E} + \frac{\hbar^2 k^2}{2m^*_{ec}} \right) a_k a_k^\dagger + \frac{\hbar^2 k^2}{2m^*_{hv}} \right] b_{-k}^\dagger b_{-k} - \left[ g_k a_k b_{-k} + g_k^* a_{-k}^\dagger b_{-k} - 1 \right] \right] \{ \text{VII-29} \}

Thus, the equation of motion for the effective dipole operator $b_{-k}^\dagger a_k$ is given by

$$\frac{d}{dt} b_{-k}^\dagger a_k = i\hbar \left[ \mathcal{H}_{\text{eff}}, b_{-k}^\dagger a_k \right] = -\left( \gamma + i\omega_k \right) b_{-k}^\dagger a_k + ig_k a_k \left( a_k^\dagger a_k + b_{-k}^\dagger b_{-k} - 1 \right) + f_{b_{-k}^\dagger a_k} \{ \text{VII-30} \}

where $\gamma$ is the damping, $\omega_k = \mathcal{E} + \frac{\hbar^2 k^2}{2m^*_{ec}} + \frac{\hbar^2 k^2}{2m^*_{hv}}$, and $f_{b_{-k}^\dagger a_k}$ is the k-dependent noise operator. We transform the various operators to a frame rotating at the electromagnetic field frequency $\omega_r$ -- viz.

$$b_{-k}^\dagger a_k = \sigma_k^* \exp[-\omega_r t] \{ \text{VII-31a} \}
$$

$$a(t) = A(t) \exp[-\omega_r t] \{ \text{VII-31b} \}
$$

$$f_{b_{-k}^\dagger a_k} = f_{\sigma_k} = F_{\sigma_k} \exp[-\omega_r t] \{ \text{VII-31c} \}
$$

and find the transformed equation of motion in the rotating frame -- i.e. the Heisenberg interaction picture -- as

$$\frac{d}{dt} \sigma_k = -\left( \gamma + i\omega_k - i\omega_r \right) \sigma_k + ig_k A \left( a_k^\dagger a_k + b_{-k}^\dagger b_{-k} - 1 \right) + F_{\sigma_k} \{ \text{VII-32} \}$$

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Applying the generalized Einstein relation -- i.e. Equation [ VII-28 ] -- to the operator

$$\sigma_k^\dagger \sigma_k = a_k^\dagger b_k^\dagger a_k b_k \equiv \sigma_{33}^k$$

we obtain

$$2 D_{\sigma_k^\dagger \sigma_k} = (\gamma + i \omega_k - i \omega_r) \sigma_{33}^k + (\gamma - i \omega_k + i \omega_r) \sigma_{33}^k \sigma_{33}^k + \sigma_{33}^k$$

$$\approx 2 \gamma \sigma_{33}^k$$  [ VII-33a ]

Similarly,

$$2 D_{\sigma_k^\dagger \sigma_k} = (\gamma + i \omega_k - i \omega_r) \sigma_{00}^k + (\gamma - i \omega_k + i \omega_r) \sigma_{00}^k + \sigma_{00}^k$$

$$\approx 2 \gamma \sigma_{00}^k$$  [ VII-33b ]

where

$$\sigma_{00}^k = (1 - a_k^\dagger a_k)(1 - b_k^\dagger b_k).$$

The equation of motion for the laser field operator in a frame rotating at the laser frequency $\omega_r$ may be written as

$$\dot{A}(t) = -\left[ \frac{\omega}{2} + i(\Omega - \omega_r) \right] A(t) - i \sum_k g_k^* \sigma_k(t) + F(t)$$  [ VII-34 ]

where $\Omega$ is the cold cavity resonant frequency and where we have replace the damping parameter $\gamma$ in equation [ VII-10 ] by the experimental parameter $\omega Q$. If we consider the Equation [ VII-32 ] in the quasi-equilibrium limit, we find

$$\sigma_k(t) = \left[ \gamma + i \omega_k - i \omega_r \right]^{-1} \left[ i g_k A(t) \left( a_k^\dagger a_k + b_k^\dagger b_k - 1 \right) + F_k(t) \right]$$

$$= D(\omega_k - \omega_r; \gamma) \left[ i g_k A(t) \left( \sigma_{33}^k - \sigma_{00}^k \right) + F_k(t) \right]$$  [ VII-35 ]
where $\mathcal{D}(\omega_k - \omega, \gamma)$ is the complex Lorentzian denominator. Thus in the quasi-equilibrium limit, Equation [ VII-34 ] becomes

$$\dot{A}(t) = -\left[ \frac{\omega}{2Q} + i(\Omega - \omega_r) - \sum_k |g_k|^2 \mathcal{D}(\omega_k - \omega, \gamma) \left( \sigma_{33}^k - \sigma_{00}^k \right) \right] A(t) + F_\lambda(t) \quad \text{[ VII-36 ]}$$

where

$$F_\lambda(t) = F(t) - i \sum_k g_k^* \mathcal{D}(\omega_k - \omega, \gamma) F_{\lambda_k}(t). \quad \text{[ VII-37 ]}$$

Following arguments similar to those used in deriving Equation [ VII-16 ], we easily derive an equation of motion for the laser photon number operator -- viz.

$$\frac{d}{dt} \langle A^\dagger(t) A(t) \rangle = -\left[ \frac{\omega}{Q} + \frac{2}{\gamma} \sum_k |g_k|^2 \mathcal{L}(\omega_k - \omega, \gamma) \left( \sigma_{33}^k - \sigma_{00}^k \right) \right] \langle A^\dagger(t) A(t) \rangle$$

$$+ \int_{-\infty}^{+\infty} d\tau \left\{ F_\lambda^*(0), F_\lambda(\tau) \right\} \quad \text{[ VII-38a ]}$$

or

$$\frac{d}{dt} \langle A^\dagger(t) A(t) \rangle = -\left[ \frac{\omega}{Q} - \left( R_{33} - R_{\lambda_0} \right) \right] \langle A^\dagger(t) A(t) \rangle + \int_{-\infty}^{+\infty} d\tau \left\{ F_\lambda^*(0), F_\lambda(\tau) \right\} \quad \text{[ VII-38b ]}$$

where

$$R_{33} = \frac{2}{\gamma} \sum_k |g_k|^2 \mathcal{L}(\omega_k - \omega, \gamma) \sigma_{33}^k. \quad \text{[ VII-39a ]}$$

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45 It may be recalled that in the notes entitled *Lasers: Models and Theories* for convenience we introduced the Lorentzian functions -- viz.

$$\mathcal{D}(u - v, w) \equiv \left[ i(u - v) + w \right]^{-1} \quad \text{the complex Lorentzian denominator}$$

$$\mathcal{L}(u - v, w) \equiv w^2 \left[ (u - v)^2 + w^2 \right]^{-1} \quad \text{the dimensionless Lorentzian function.}$$

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and

\[ R_{00} = \frac{2}{\gamma} \sum_{k} |g_k|^2 \mathcal{L}(\omega_k - \omega, \gamma) \sigma_{00}^{k}. \]  \[ \text{[VII-39b]} \]

Assuming that the cavity damping and carrier scattering reservoirs are uncorrelated, the two-time correlation function may be written as

\[ \begin{align*}
\{ F_A^\dagger(t) F_A(t') \} &= \{ F(t) F(t') \} \\
&+ \sum_{k} \sum_{k'} g_k g_k^* \mathcal{D}(\omega_k - \omega, \gamma) \mathcal{D}(\omega_{k'} - \omega, \gamma) \{ F_{\sigma_k}(t) F_{\sigma_{k'}}(t') \} \end{align*} \]  \[ \text{[VII-40a]} \]

and if the noise operators for different \( \mathbf{k} \) are uncorrelated

\[ \{ F_A^\dagger(t) F_A(t') \} = \{ F(t) F(t') \} + \frac{1}{\gamma} \sum_{k} |g_k|^2 \mathcal{L}(\omega_k - \omega, \gamma) \{ F_{\sigma_k}(t) F_{\sigma_k}(t') \}. \]  \[ \text{[VII-40b]} \]

Finally, using Equations \[ \text{[VII-19]} \] and \[ \text{[VII-33a]} \] we obtain

\[ \begin{align*}
\{ F_A^\dagger(t) F_{\sigma_k}(t') \} &= \left[ \frac{\omega}{Q} \{ n(\omega) \}_{\text{therm}} + \frac{2}{\gamma} \sum_{k} |g_k|^2 \mathcal{L}(\omega_k - \omega, \gamma) \sigma_{33}^{k} \right] \delta(t - t'). \\
&\text{[VII-40c]} \end{align*} \]

and, thus, the "\( A^\dagger A \)" diffusion coefficient for the laser field is

\[ 2 D_{A^\dagger A} = \frac{\omega}{Q} \{ n(\omega) \}_{\text{therm}} + R_{33}. \]  \[ \text{[VII-41a]} \]

By a similar argument, the "\( AA^\dagger \)" diffusion coefficient is

\[ 2 D_{AA^\dagger} = \frac{\omega}{Q} \left( \{ n(\omega) \}_{\text{therm}} + 1 \right) + R_{00}. \]  \[ \text{[VII-41b]} \]

Using Equation \[ \text{[VII-41a]} \], Equation \[ \text{[VII-38b]} \] becomes
\[
\frac{d}{dt}\{A(t), A(t)\} = -\left[\frac{\omega}{Q} - (R_{33} - R_{00})\right]\{A(t), A(t)\} + \frac{\omega}{Q} \{n(\omega)\}_{\text{therm}} + R_{33} \quad [\text{VII-42a}]
\]

Similarly
\[
\frac{d}{dt}\{A(t), A^\dagger(t)\} = -\left[\frac{\omega}{Q} - (R_{33} - R_{00})\right]\{A(t), A^\dagger(t)\} + \frac{\omega}{Q} \{n(\omega)\}_{\text{therm}} + 1 + R_{00} \quad [\text{VII-42b}]
\]

As check on the consistency of these result, we see that the form of Equations [ VII-42a ] and [ VII-42b ] guarantees that the commutator \([A(t), A^\dagger(t)]\) is independent of time -- viz.
\[
\frac{d}{dt}\{A(t), A^\dagger(t)\} = -\omega Q - R_{33} - R_{00} \left[\frac{\omega}{Q}\right] A(t), A^\dagger(t) + \omega Q + R_{00} - R_{33} = 0 \quad [\text{VII-43}]
\]

to incorporate saturation effects into the noise spectrum we need to write a Langevin equation for the carrier-density operator -- viz.
\[
\frac{d}{dt}a^\dagger_k a_k = \frac{i}{\hbar} [\mathcal{H}_{\text{eff}}, a^\dagger_k a_k] + \{\text{damping, pumping, and noise terms}\}
\]
\[
= \Lambda e \left[1 - a^\dagger_k a_k\right] - \Gamma_k a^\dagger_k a_k \hat{b}^\dagger_{-k} \hat{b}_{-k} - \gamma_{nr} a^\dagger_k a_k
\]
\[
- \gamma_e \left[a^\dagger_k a_k - \left(a^\dagger_k a_k\right)_{\text{eq}}\right] - i \left[g_k a^\dagger_k \hat{b}^\dagger_{-k} a - g^*_k a^\dagger_k \hat{b}_{-k} a\right] + F_e^k
\]

where \(\Lambda e \left[1 - a^\dagger_k a_k\right]\) is the pumping due injected carriers, \(\Gamma_k a^\dagger_k a_k \hat{b}^\dagger_{-k} \hat{b}_{-k}\) is the radiative recombination (spontaneous emission) term, \(\gamma_{nr} a^\dagger_k a_k\) is the nonradiative recombination term, \(\gamma_e \left[a^\dagger_k a_k - \left(a^\dagger_k a_k\right)_{\text{eq}}\right]\) is the relaxation due carrier-carrier scattering, and \(F_e^k\) is the carrier noise operator. In the quasi-equilibrium approximation -- i.e. from Equation [ VII-35 ] -- we may sum Equation III-44 to obtain a Langevin equation for the total carrier-density operator.
\[ \frac{d}{dt} N = \Lambda - \frac{1}{V} \sum_k \Gamma_k a_k^\dagger a_k^\dagger b_k^\dagger b_k - \gamma_{nr} N \]

\[ - \frac{2A^\dagger A}{\gamma V} \sum_k |g_k|^2 \mathcal{L}(\omega_k - \omega_r, \gamma) \left( \sigma_{33}^k - \sigma_{00}^k \right) + F_N \]

where

\[ F_N(t) = F_e^k(t) + \frac{i}{V} \sum_k \left[ g_k^* A^\dagger(t) \mathcal{D}(\omega_k - \omega_r, \gamma) F_{\sigma_k}(t) - g_k A(t) \mathcal{D}^\dagger(\omega_k - \omega_r, \gamma) F_{\sigma_k}^\dagger(t) \right] \]

**NOISE SPECTRA:**

To calculate noise spectra we make use of semiclassical transformations -- viz.

\[ A(t) = \left( E(t)/\mathcal{E}_{\omega_0} \right) \exp[-i\phi(t)] \]

\[ A^\dagger(t) = \left( E(t)/\mathcal{E}_{\omega_0} \right) \exp[i\phi(t)] \]

\[ E(t) = \mathcal{E}_{\omega_0} \sqrt{A^\dagger(t) A(t)} \]

\[ \phi(t) = (i/2) \ln[A(t)/A^\dagger(t)] \]

where \( \mathcal{E}_{\omega_0} = [\hbar \omega, \mathcal{E}_0 V] \) is the electric field per photon. Using the expression

\[ D^\rho_{\mu \nu} = \sum_{\mu, \nu} D^\rho_{\mu \nu} \frac{\partial A^\dagger_{\mu} \partial A^\dagger_{\nu}}{\partial A^\rho_{\mu} \partial A^\rho_{\nu}} \]

we see that
\[ D_{EE} = D_{A^\dagger A} \frac{\partial E}{\partial A^\dagger} \frac{\partial E}{\partial A} + D_{AA^\dagger} \frac{\partial E}{\partial A} \frac{\partial E}{\partial A^\dagger} \]

\[ = \frac{\mathcal{E}_\omega^4}{4E^2} \left[ AA^\dagger D_{A^\dagger A} + A^\dagger A D_{AA^\dagger} \right] = \frac{\mathcal{E}_\omega^4}{4} \left[ D_{A^\dagger A} + D_{AA^\dagger} \right] \]

\[ D_{\phi\phi} = D_{A^\dagger A} \frac{\partial \phi}{\partial A^\dagger} \frac{\partial \phi}{\partial A} + D_{AA^\dagger} \frac{\partial \phi}{\partial A} \frac{\partial \phi}{\partial A^\dagger} \]

\[ = \frac{1}{4AA^\dagger} D_{A^\dagger A} + \frac{1}{4A^\dagger A} D_{AA^\dagger} = \frac{\mathcal{E}_\omega^2}{4E^2} \left[ D_{A^\dagger A} + D_{AA^\dagger} \right] \]