ANTENNAS: A GENERAL FORMULATION IN FAR-FIELD

I. THE FORMULATION:

In general, the fields radiated by an antenna can be obtained by evaluating the previously discussed general formula for the (magnetic) vector field -- *viz*.

$$\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega) = \frac{\mu}{4\pi}\iiint \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\exp\left[-j\,k\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|\right]}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|}\,dV' \qquad \text{[I-1]}$$

Differentiation of the vector potential thus obtained yields complete expressions for the fields in the near, intermediate and far-field regions. However, in most instances, **this is more information than is needed in most applications of the theory**. If one is interested **only** in the **far-field** characteristics of the antenna, there is a simpler, more direct approach. This approach is ordinarily more than adequate. From previous discussion, we know that the far-fields of an elemental Hertzian dipole are given by

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = -j\,\frac{k\;I(\omega)\,\delta\,\ell}{4\pi}\,\frac{\exp(-j\;k\,r)}{r}\left[\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right] \qquad \text{[I-2a]}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = j\,\frac{\eta\;k\;I(\omega)\,\delta\,\ell}{4\pi}\,\frac{\exp(-j\;k\,r)}{r}\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\hat{\mathbf{z}}\right]\right\} \qquad \text{[I-2b]}$$

Since Maxwell's equations are linear, we may obtain the **far-fields** of a general antenna by summing the elemental far-field contributions of each current element of the antenna treated as an elemental Hertzian dipole -- *i.e*.

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = -j\,\frac{k}{4\pi}\iiint_{\text{antenna}}\frac{\exp(-j\;k\,R)}{R}\left[\hat{\mathbf{R}}\times\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\right]dV' \qquad \text{[I-3a]}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = j\,\frac{\eta\;k}{4\pi}\iiint_{\text{antenna}}\frac{\exp(-j\;k\,R)}{R}\left\{\hat{\mathbf{R}}\times\left[\hat{\mathbf{R}}\times\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\right]\right\}dV' \qquad \text{[I-3b]}$$

where

$$\vec{\mathbf{R}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}' \qquad \text{[I-4a]}$$

and
$$R = \left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right| = \sqrt{r^2 + r'^2 - 2\,\vec{\mathbf{r}}\ \vec{\mathbf{r}}'} = r\sqrt{1 + \frac{r'^2}{r^2} - \frac{2\,\vec{\mathbf{r}}\ \vec{\mathbf{r}}'}{r^2}}\ . \qquad \text{[I-4b]}$$

To obtain a suitable approximation at distances far from the antenna (*i.e.* for $r'/r << 1$), we use the **binomial expansion** -- *viz.*

$$R = \left|\vec{\mathbf{r}} - \vec{\mathbf{r}}'\right| \approx r\left(1 - \frac{\vec{\mathbf{r}}\ \vec{\mathbf{r}}'}{r^2}\right) = r - \vec{\mathbf{r}}\ \vec{\mathbf{r}}'/r = r - r'\cos\Psi\ . \qquad \text{[I-5]}$$

where $\cos\Psi \equiv \hat{\mathbf{r}}\ \hat{\mathbf{r}}'$. In approximating the the "phase sensitive" exponential factor we must insure that changes in $k\,R$ over the antenna volume are much less than π so both terms are included in that exponential factor, otherwise we may take $R \approx r$. Thus, the far fields are given by

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = -j\,\frac{k}{4\pi}\,\frac{\exp(-j\,k\,r)}{r}\left[\hat{\mathbf{r}} \times \vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[I-6a]}$$

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = j\,\frac{\eta\,k}{4\pi}\,\frac{\exp(-j\,k\,r)}{r}\left\{\hat{\mathbf{r}} \times \left[\hat{\mathbf{r}} \times \vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right]\right\} \qquad \text{[I-6a]}$$

where

$$\vec{\mathbf{N}}(\vec{\mathbf{r}},\omega) = \iiint_{\text{antenna}} \vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\exp\left[j\,k\,\vec{\mathbf{r}}'\ \hat{\mathbf{r}}\right]dV' = \iiint_{\text{antenna}} \vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\exp\left[j\,k\,r'\cos\Psi\right]dV' \qquad \text{[I-7]}$$

is a kind of ***Fourier transform*** of the current distribution on the antenna and is the key element in various applications of this formulation. The most valuable application is the resulting expression for the **Poynting vector** in the far-field, to wit

$$\vec{\mathbf{S}}(\vec{\mathbf{r}},\omega) = \frac{1}{2}\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\times\vec{\mathbf{H}}^{*}(\vec{\mathbf{r}},\omega)$$

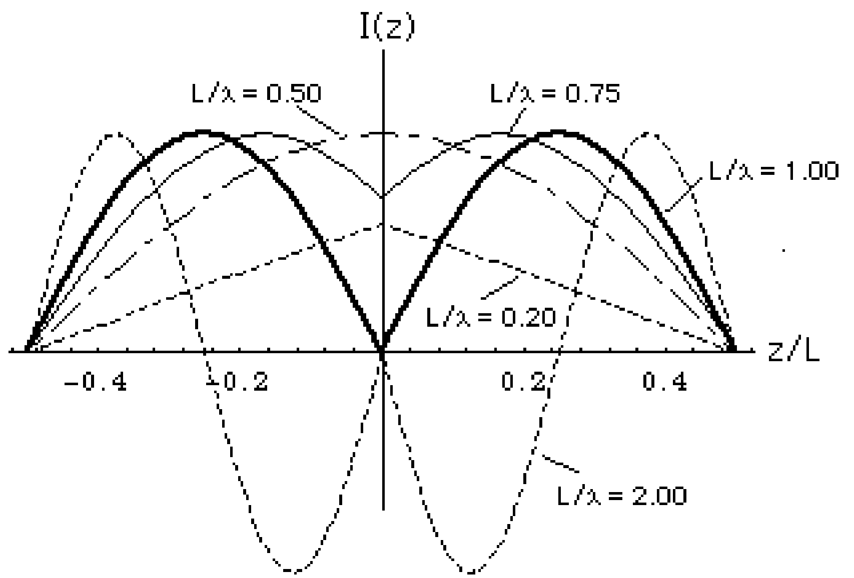
$$= \frac{1}{2}\left\{ j\,\frac{\eta\,k}{4\pi}\,\frac{\exp(-j\,k\,r)}{r}\left\{\hat{\mathbf{r}}\times\left[\hat{\mathbf{r}}\times\vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right]\right\}\right\}\times\left\{ j\,\frac{k}{4\pi}\,\frac{\exp(j\,k\,r)}{r}\left[\hat{\mathbf{r}}\times\vec{\mathbf{N}}^{*}(\vec{\mathbf{r}},\omega)\right]\right\} \qquad \text{[I-8]}$$

$$= \frac{\eta\,k^2}{32\pi^2}\left|\hat{\mathbf{r}}\times\vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right|^2\frac{\hat{\mathbf{r}}}{r^2} = \frac{\eta}{8\,\lambda^2}\left|\hat{\mathbf{r}}\times\vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right|^2\frac{\hat{\mathbf{r}}}{r^2}$$

II. IMPORTANT APPLICATION -- A linear, filamentary antenna with <u>standing</u> <u>wave</u> excitation:

Consider a current distribution on a center-excited, straight filamentary (very, very thin) wire which may be approximated as a **standing wave** -- *viz.* suppose that

$$\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\,dV' = \hat{\mathbf{z}}\,I_0\,\sin\left[\beta\left(\frac{L}{2}-|z'|\right)\right]dz' \qquad \text{for} \qquad 0\le|z'|\le L/2\;. \qquad \text{[II-1]}$$



EXAMPLES OF STANDING WAVE CURRENT DISTRIBUTIONS

Given this **assumed current distribution**, we have the following current transform:

$$\vec{\mathbf{N}}(\theta,\omega) = \hat{\mathbf{z}}\, I_0 \left\{ \int_{-L/2}^{0} \left[\sin\left(\frac{\beta L}{2} + \beta z'\right) \exp(j\, k\, z' \cos\theta)\, dz' \right] + \int_{0}^{+L/2} \left[\sin\left(\frac{\beta L}{2} - \beta z'\right) \exp(j\, k\, z' \cos\theta)\, dz' \right] \right\} \qquad \text{[II-2]}$$

Using the standard form

$$\int \exp(\mathrm{A}\, x) \sin(\mathrm{B}\, x)\, dx = \exp(\mathrm{A}\, x) \left[\frac{\mathrm{A} \sin(\mathrm{B}\, x) - \mathrm{B} \sin(\mathrm{B}\, x)}{\mathrm{A}^2 + \mathrm{B}^2} \right] \qquad \text{[II-3]}$$

we obtain

$$\vec{\mathbf{N}}(\theta,\omega) = \hat{\mathbf{z}}\, 2\, I_0\, \beta \left[\frac{\cos\left(\frac{k L \cos\theta}{2}\right) - \cos\left(\frac{\beta L}{2}\right)}{\beta^2 - k^2 \cos^2\theta} \right]. \qquad \text{[II-4]}$$

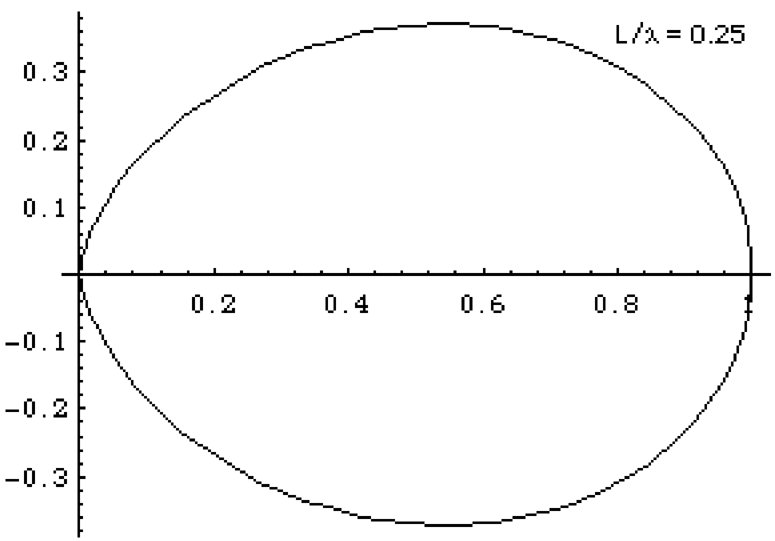
Substituting this transform into Equation [I-8], we obtain

$$\begin{aligned} \vec{\mathbf{S}}(\vec{\mathbf{r}},\omega) &= \frac{\eta}{8\,\lambda^2} \left| \hat{\mathbf{r}} \times \vec{\mathbf{N}}(\vec{\mathbf{r}},\omega) \right|^2 \frac{\hat{\mathbf{r}}}{r^2} \\ &= \frac{\eta}{8\,\lambda^2} \left| I_0 \right|^2 \left\{ \frac{2\,\beta \sin\theta}{\beta^2 - k^2 \cos^2\theta} \left[\cos\left(\frac{k L \cos\theta}{2}\right) - \cos\left(\frac{\beta L}{2}\right) \right] \right\}^2 \frac{\hat{\mathbf{r}}}{r^2} . \end{aligned} \qquad \text{[II-5]}$$

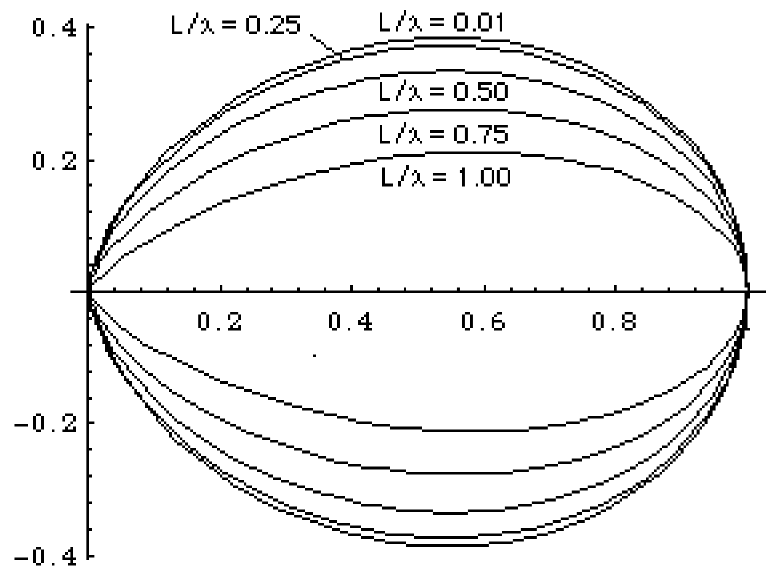
For **TEM** excitation -- *i.e.* $\beta \equiv k$

$$\vec{\mathbf{S}}(\vec{\mathbf{r}},\omega)=\frac{\eta}{8\,\pi^2}\,|I_0|^2\left[\frac{\cos\left(\frac{k\,L\cos\theta}{2}\right)-\cos\left(\frac{\beta\,L}{2}\right)}{\sin\theta}\right]^2\frac{\hat{\mathbf{r}}}{r^2}\,. \qquad \text{[II-6]}$$

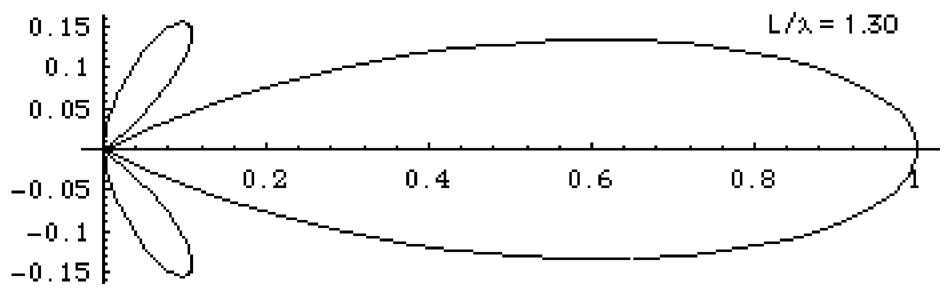
Let us examine the angular variation (**radiation pattern**) of this rather complicated expression as a function of **antenna length**.



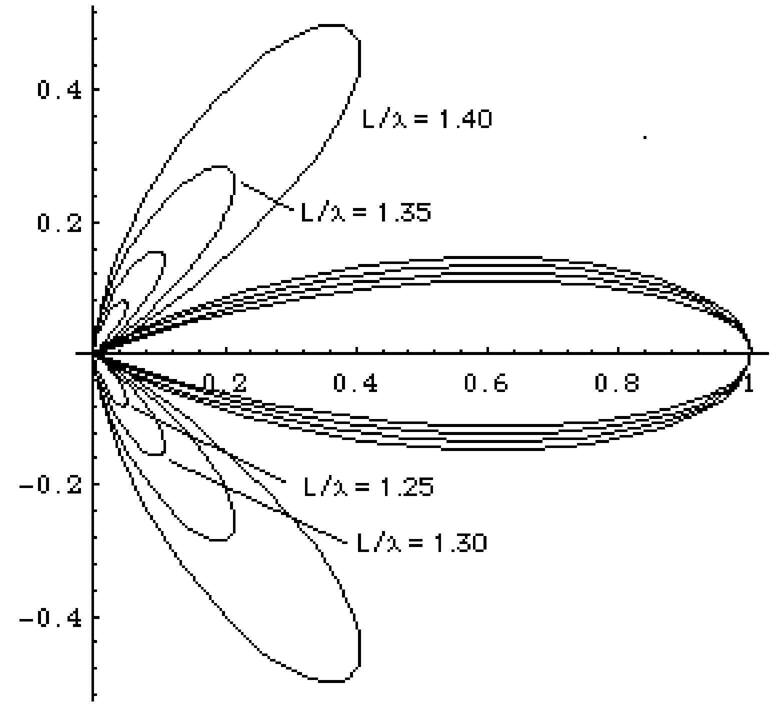
Normalized radiation pattern for length =wavelength/4



Normalized radiation pattern for various lengths (λ/100, λ/4, λ/2, 3λ/4, and λ)



Normalized radiation pattern for length =1.30 wavelength



Normalized radiation pattern for various lengths (1.25λ, 1.30λ, 1.35λ, and 1.40λ)

III. IMPORTANT APPLICATION -- A linear, filamentary antenna with <u>traveling</u> <u>wave</u> excitation:

Consider a current distribution on an end-excited, straight filamentary (very, very thin) wire which may be approximated as a **traveling wave** -- *viz.* suppose that

$$\vec{\mathbf{J}}(\vec{\mathbf{r}}',\omega)\, dV' = \hat{\mathbf{z}}\, I_0 \exp(-j\beta\, z')\, dz' \qquad \text{for} \qquad 0 \le z' \le L \; . \qquad \text{[III-1]}$$

Given this assumed current distribution we now have the following current transform:

$$\begin{aligned}\vec{\mathbf{N}}(\theta,\omega) &= \hat{\mathbf{z}}\, I_0 \int_0^L \left[\exp(-j\,\beta\, z')\,\exp(j\,k\,z'\,\cos\theta)\,dz'\right] \\ &= \hat{\mathbf{z}}\, I_0 \int_0^L \exp\left[j\,(k\cos\theta-\beta)\,z'\right] dz'\end{aligned} \qquad \text{[III-2]}$$

Carrying out the elementary integral, we obtain

$$\vec{\mathbf{N}}(\theta,\omega) = \hat{\mathbf{z}}\, I_0\, L\, \exp\left[j\,(k\cos\theta-\beta)\frac{L}{2}\right] \mathrm{sinc}\left[(k\cos\theta-\beta)\frac{L}{2}\right]. \qquad \text{[III-3]}$$

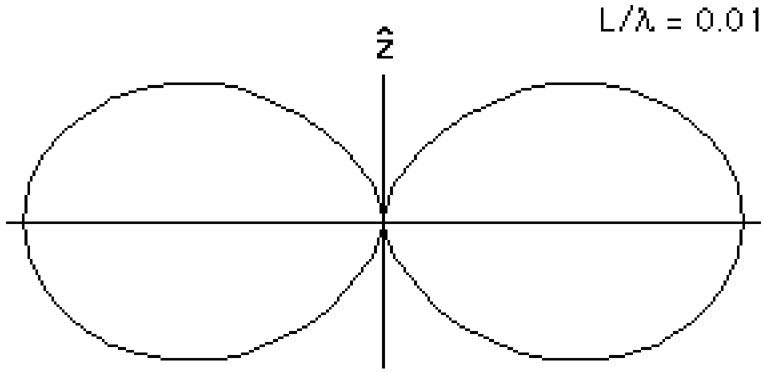
Substituting this transform into Equation [I-8], we obtain

$$\begin{aligned}\vec{\mathbf{S}}(\vec{\mathbf{r}},\omega) &= \frac{\eta\, k^2}{32\pi^2}\left|\hat{\mathbf{r}}\times\vec{\mathbf{N}}(\vec{\mathbf{r}},\omega)\right|^2 \frac{\hat{\mathbf{r}}}{r^2} \\ &= \frac{\eta}{32\pi^2}\left|I_0\right|^2 \left\{k\,L\,\sin\theta\,\mathrm{sinc}\left[(k\cos\theta-\beta)\frac{L}{2}\right]\right\}^2 \frac{\hat{\mathbf{r}}}{r^2}.\end{aligned} \qquad \text{[III-4]}$$

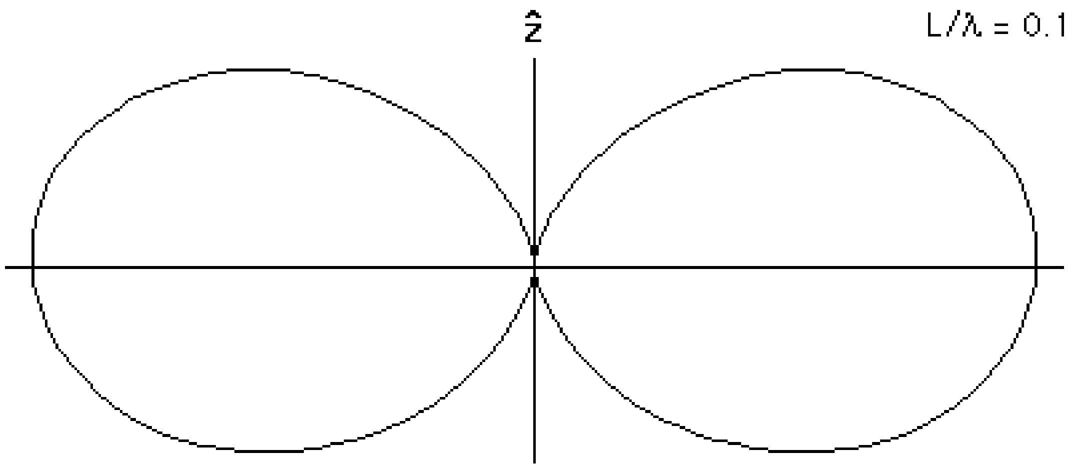
For **TEM** excitation -- *i.e.* $\beta \equiv k$

$$\vec{\mathbf{S}}(\vec{\mathbf{r}},\omega) = \frac{\eta}{32\pi^2}\left|I_0\right|^2 \left\{k\,L\,\sin\theta\,\mathrm{sinc}\left[k\,L\,\sin^2(\theta/2)\right]\right\}^2 \frac{\hat{\mathbf{r}}}{r^2}. \qquad \text{[III-5]}$$

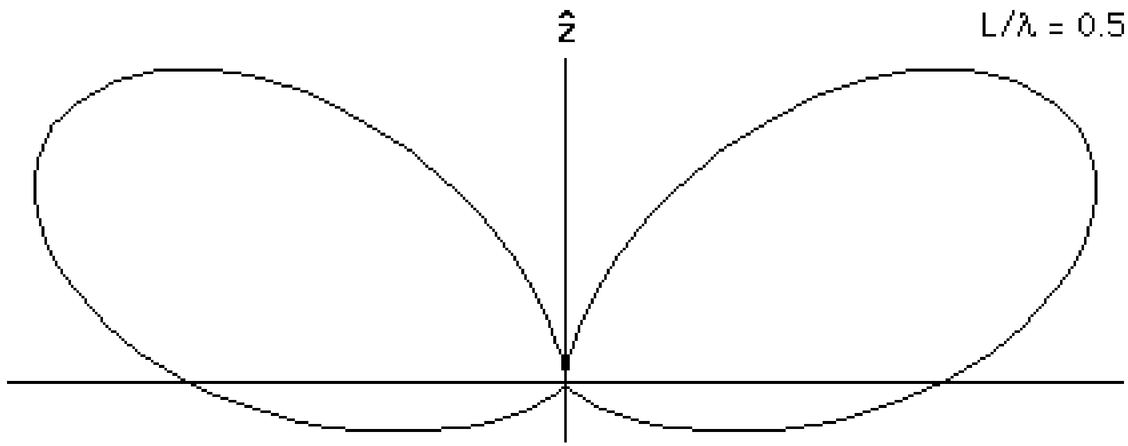
Let us examine the angular variation (**radiation pattern**) of this expression as a function of **antenna length**.



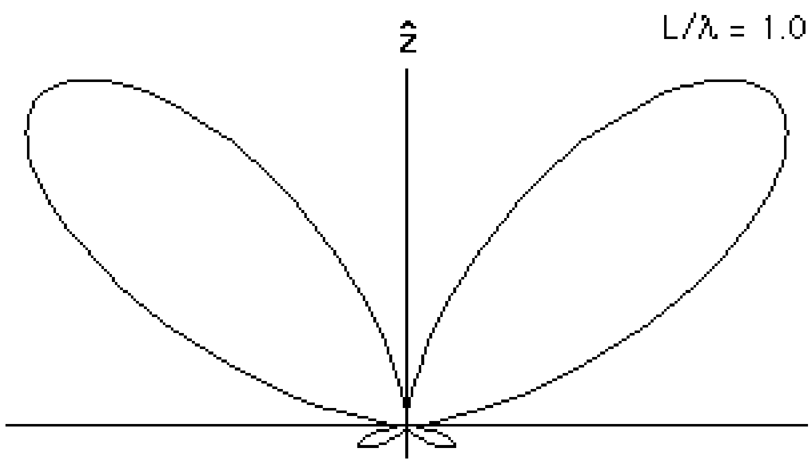
Radiation pattern for antenna length = 0.01 times wavelength

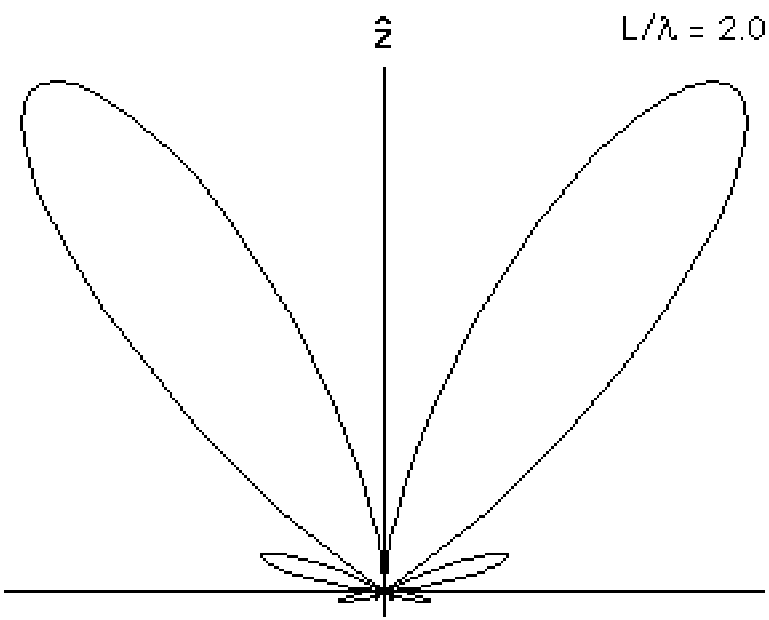


Radiation pattern for antenna length = 0.1 times wavelength

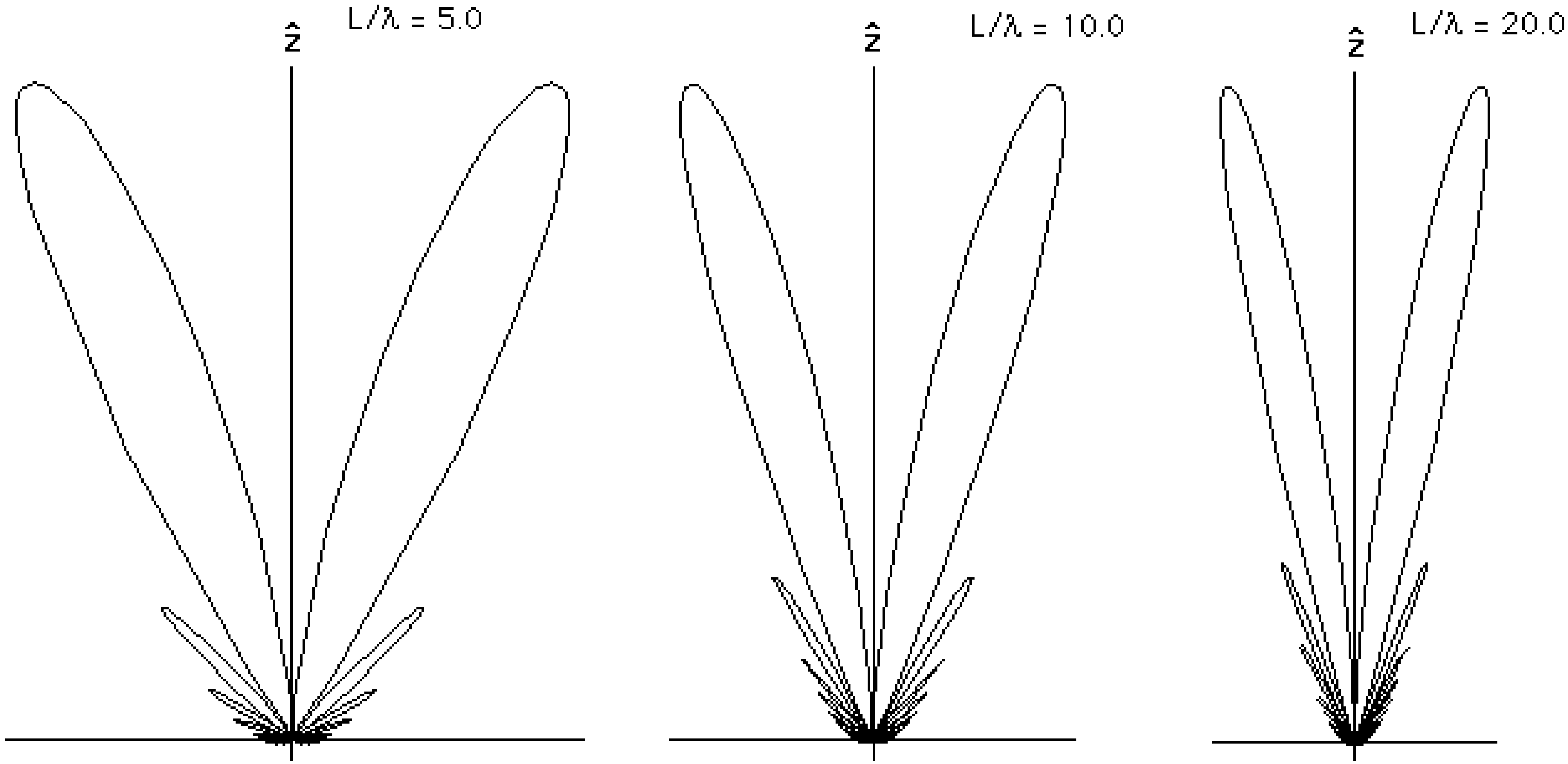


Radiation pattern for antenna length = 0.5 times wavelength





Radiation patterns for antenna lengths = 1.0 and 2.0 times wavelength



Radiation patterns for antenna lengths = 5, 10 and 20 times wavelength

The critical point is that the power flow peaks at lobes defined by

$$\frac{kL}{2}\left(\frac{\beta}{k} - \cos\theta_m\right) \equiv \left(\frac{2m+1}{2}\right)\pi \qquad \text{[III-6]}$$

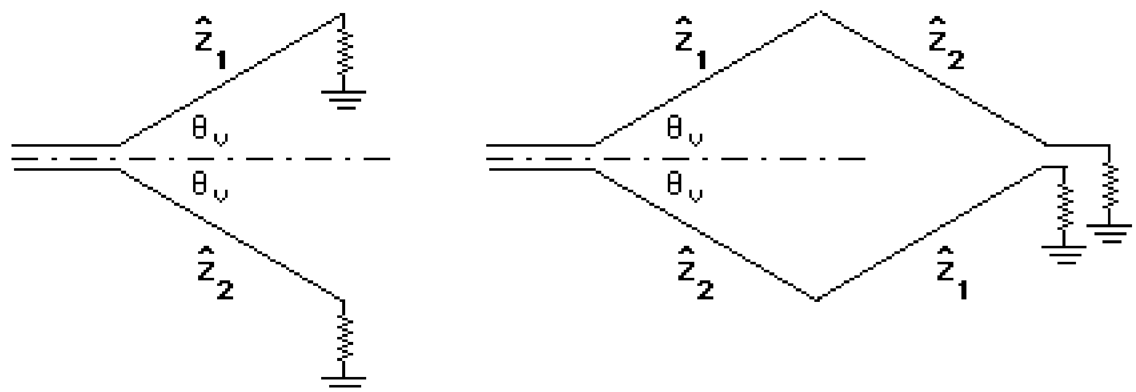
and falls to zero at nulls defines by

$$\frac{kL}{2}\left(\frac{\beta}{k} - \cos\theta_n\right) \equiv n\,\pi \qquad \text{[III-7]}$$

where m and n are integers. The major peak or lobe is given by $m = 0$ so that

$$\theta_{\max} = a\cos\left(\frac{\beta}{k} - \frac{\pi}{k\,L}\right) \approx a\cos\left(1 - \frac{\pi}{k\,L}\right) = a\cos\left(1 - \frac{\lambda}{2\,L}\right) \qquad \text{[III-8]}$$

Two exceeding useful radiating structures -- *viz.* **"V" and "rhombic" antennas** -- may be obtained by arraying single filament traveling wave antennas as illustrated in the following figure:



A V-ANTENNA **A RHOMBIC-ANTENNA**

Quite generally we may write

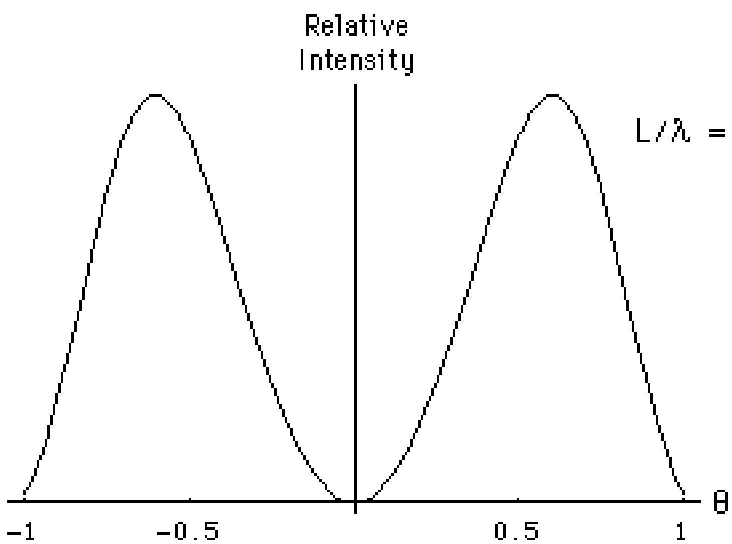
$$\hat{\mathbf{z}}_1 = \sin\theta_v\,\hat{\mathbf{x}} + \cos\theta_v\,\hat{\mathbf{z}}$$
$$\hat{\mathbf{r}} = \sin\theta\,\sin\varphi\,\hat{\mathbf{x}} + \sin\theta\,\cos\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\cos\theta_1 = \sin\theta_v\,\sin\theta\,\sin\varphi + \cos\theta_v\,\cos\theta$$
$$\cos\theta_2 = -\sin\theta_v\,\sin\theta\,\sin\varphi + \cos\theta_v\,\cos\theta$$
$$\hat{\mathbf{r}} \times \hat{\mathbf{z}}_1 = \left(\cos\theta\,\sin\theta_v - \sin\theta\,\sin\varphi\cos\theta_v\right)\hat{\mathbf{y}} - \sin\theta\,\cos\varphi\,\sin\theta_v\,\hat{\mathbf{z}} + \sin\theta\,\cos\varphi\cos\theta_v\,\hat{\mathbf{x}}$$
$$\hat{\mathbf{r}} \times \hat{\mathbf{z}}_2 = \left(-\cos\theta\sin\theta_v - \sin\theta\,\sin\varphi\,\cos\theta_v\right)\hat{\mathbf{y}} + \sin\theta\,\cos\varphi\sin\theta_v\,\hat{\mathbf{z}} + \sin\theta\,\cos\varphi\,\cos\theta_v\,\hat{\mathbf{x}}$$

In the "V-plane" -- *i.e.* for $\varphi = \pi/2$ -- these expressions simplify to

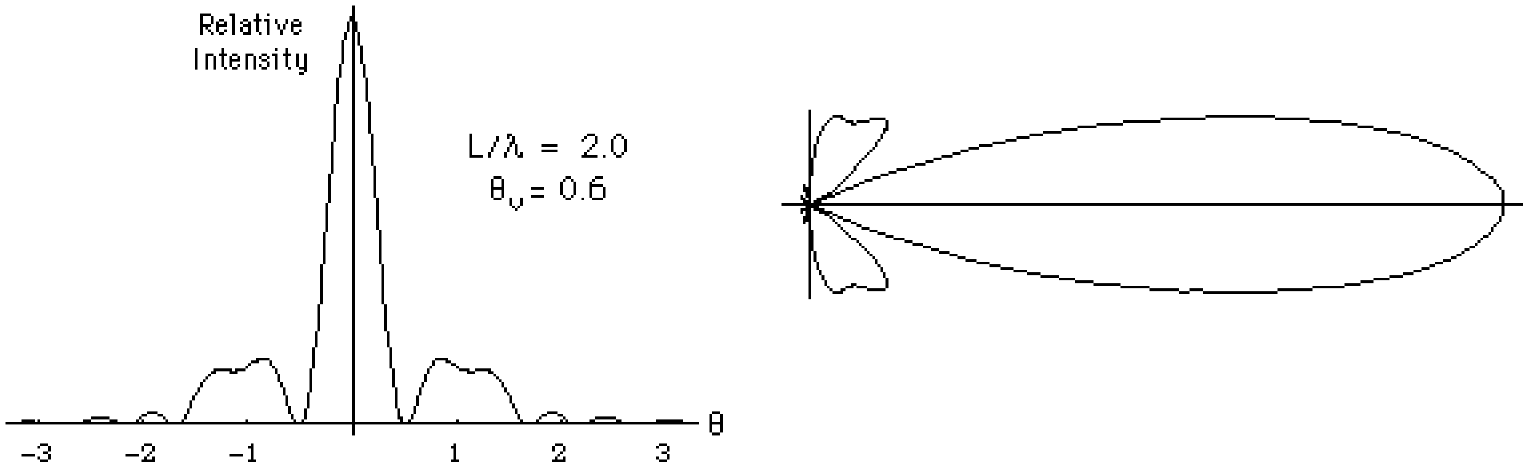
$$\hat{\mathbf{z}}_1 = \sin\theta_v\ \hat{\mathbf{x}} + \cos\theta_v\ \hat{\mathbf{z}}$$
$$\hat{\mathbf{r}} = \sin\theta\ \hat{\mathbf{x}} + \cos\theta\ \hat{\mathbf{z}}$$
$$\cos\theta_1 = \cos(\theta - \theta_v)$$
$$\cos\theta_2 = \cos(\theta + \theta_v)$$
$$\hat{\mathbf{r}} \times \hat{\mathbf{z}}_1 = -\sin(\theta - \theta_v)\ \hat{\mathbf{y}}$$
$$\hat{\mathbf{r}} \times \hat{\mathbf{z}}_2 = -\sin(\theta + \theta_v)\ \hat{\mathbf{y}}$$

and Equation [III-5] becomes

$$\vec{\mathbf{S}}(\vec{\mathbf{r}},\omega) = \frac{\eta}{32\pi^2}\,|I_0|^2 \left\{ k\,L\,\sin(\theta+\theta_v)\,\mathrm{sinc}\left[k\,L\,\sin^2\left(\frac{\theta+\theta_v}{2}\right)\right] - k\,L\,\sin(\theta-\theta_v)\,\mathrm{sinc}\left[k\,L\,\sin^2\left(\frac{\theta-\theta_v}{2}\right)\right]\right\}^2 \frac{\hat{\mathbf{r}}}{r^2}\ . \qquad \text{[III-9]}$$

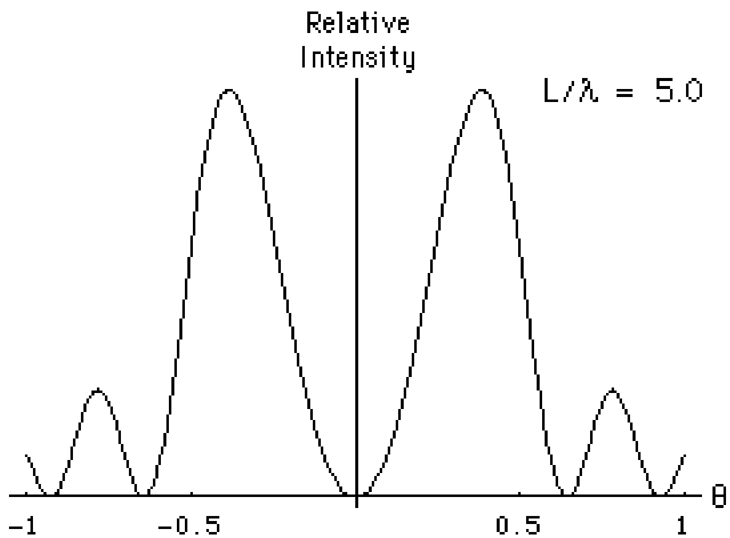


Radiation pattern of a single filamentary traveling wave antenna ($L/\lambda = 2.0$)

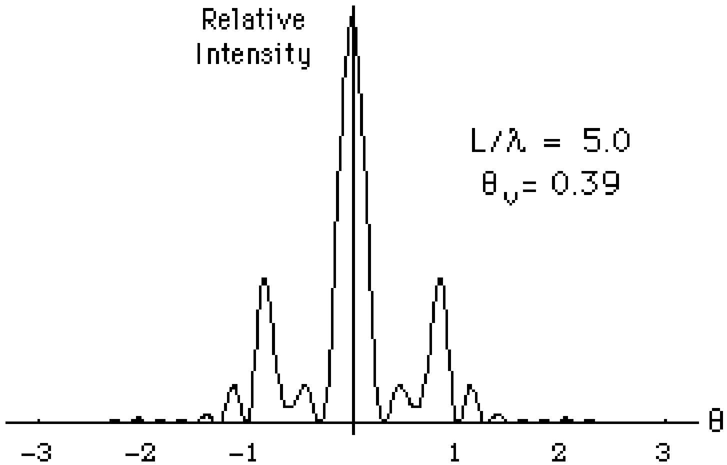


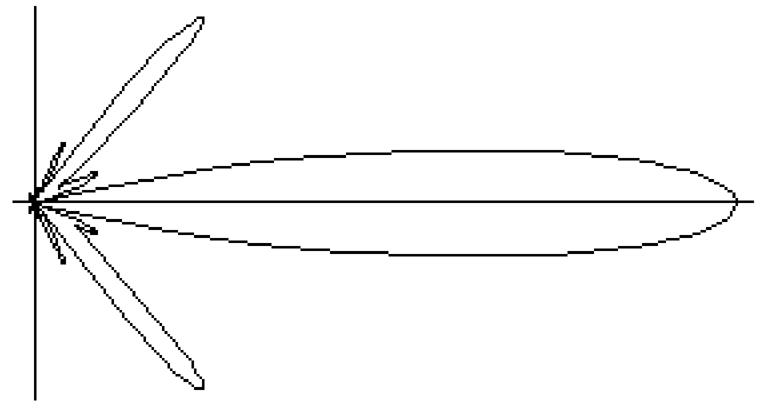
linear representation **polar representation**

Radiation pattern of corresponding V-antenna
(θ_v = 0.60 rads = 34°)



Radiation pattern of a longer single filamentary traveling wave antenna ($L/\lambda = 5.0$)





linear representation **polar representation**

Radiation pattern of corresponding V-antenna

(θ_v = 0.39 rads = 22°)