FOURIER OPTICS

I. THE DIFFRACTION INTEGRAL:

It may be fairly said that "it is well known" that plane waves are solutions of the homogeneous Helmholz equation.[1] Thus, by a generalization of the Fourier theorem, we expect to be able to write a general solution for a given component of the (magnetic) vector potential in the form

$$\mathrm{A}_\alpha\left(\vec{\mathbf{r}},\omega\right)=\iiint \mathrm{A}_\alpha\left(\vec{\mathbf{q}},\omega\right)\,\exp\!\left[-j\,\vec{\mathbf{q}}\;\;\vec{\mathbf{r}}\right]\delta\!\left(\mathrm{q}^2-k^2\right)d\,\mathrm{q}_x\,d\mathrm{q}_y\,d\,\mathrm{q}_z \qquad \text{[I-1]}$$

where the inclusion of the **Dirac delta function** insures that the Helmholz equation is satisfied.[2] Executing the integration over q_z we obtain[3]

$$\begin{aligned}\mathrm{A}^{\pm}_{\alpha}\left(\vec{\mathbf{r}},\omega\right)=\iint \mathrm{A}^{\pm}_{\alpha}\left(\mathrm{q}_x,\mathrm{q}_y,\omega\right)&\exp\!\left[-j\,\mathrm{q}_x\,x\right]\;\exp\!\left[-j\,\mathrm{q}_y\,y\right]\\ &\times\exp\!\left[\mp j\sqrt{k^2-\mathrm{q}_x^2-\mathrm{q}_y^2}\;z\right]d\,\mathrm{q}_x\,d\,\mathrm{q}_y\end{aligned}\;. \qquad \text{[I-2]}$$

Evaluating this expression in the **paraxial approximation**, leads to the expression

---

[1] For the record the homogeneous Helmholtz equation is

$${}^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+\omega^2\,\mu_0\,\varepsilon(\omega)\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)={}^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+k^2\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=0$$

[2] The Dirac delta function is defined as

$$\int_{x_0-a}^{x_0+b}\mathrm{f}(x)\,\delta\!\left(x-x_0\right)dx\equiv\mathrm{f}\!\left(x_0\right)$$

[3] Its a fussy point, but there are, of course two valid values of q_z -- *viz.* $\mp\sqrt{k^2-\mathrm{q}_x^2-\mathrm{q}_y^2}$. To remind ourselves of this slight complication, we include a ± superscript on the vector potential and other variables to account for the possibility of fields propagating in either or both the positive (upper sign) and negative (lower sign) *z*-directions.

$$A_\alpha^\pm(\vec{\mathbf{r}},\omega) \approx \exp[\mp j\,k\,z]\iint A_\alpha^\pm(q_x,q_y,\omega)\exp[-j\,q_x\,x]\;\exp[-j\,q_y\,y]\exp\left[\pm j\,\frac{q_x^2+q_y^2}{2k}\,z\right]d\,q_x\,d\,q_y \qquad [\text{I-3}]$$

If the vector potential is known over some particular plane, say the plane $z = z_0$, then

$$A_\alpha^\pm(q_x,q_y,\omega) = \left(\frac{1}{2\pi}\right)^2\exp[\pm j\,kz_0]\exp\left[\mp j\left(\frac{q_x^2+q_y^2}{2k}\right)z_0\right]\iint A_\alpha^\pm(x,y,z_0,\omega)\exp[j\,q_x\,x]\;\exp[j\,q_y\,y]\,dx\;dy \qquad [\text{I-4}]$$

Therefore, the general solution in the paraxial approximation may be written

$$\begin{aligned} A_\alpha^\pm(x,y,z,\omega) \approx \exp[\mp j\,k\,(z-z_0)]\iint\Bigg\{&\exp[-j\,q_x\,x]\exp[-j\,q_y\,y]\exp\left[\pm j\left(\frac{q_x^2+q_y^2}{2k}\right)(z-z_0)\right] \\ &\times\left(\frac{1}{2\pi}\right)^2\iint A_\alpha^\pm(x',y',z_0,\omega)\exp[j\,q_x\,x']\;\exp[j\,q_y\,y']\,d\,x'\,d\,y'\Bigg\}\,d\,q_x\,d\,q_y \end{aligned} \qquad [\text{I-5}]$$

Inverting the order of the integrations, we obtain

$$\begin{aligned} A_\alpha^\pm(x,y,z,\omega) \approx \exp[\mp j\,k\,(z-z_0)]\left(\frac{1}{2\pi}\right)^2\iint\Bigg\{&\int\exp[-j\,q_x(x-x')]\;\exp\left[\pm j\,\frac{q_x^2}{2k}\,(z-z_0)\right]d\,q_x \\ &\times\int\exp[-j\,q_y(y-y')]\;\exp\left[\pm j\,\frac{q_y^2}{2k}\,(z-z_0)\right]d\,q_y\Bigg\}A_\alpha^\pm(x',y,z_0,\omega)\,d\,x'\,d\,y' \end{aligned} \quad . \qquad [\text{I-6}]$$

Finally, we write

$$A_\alpha^\pm(x,y,z,\omega) \approx \exp[\mp j\;k\,(z-z_0)]\iint A_\alpha^\pm(x',y',z_0,\omega)\quad{}^{\pm}(x-x',z-z_0)\quad{}^{\pm}(y-y',z-z_0)\,d\,x'\,d\,y' \qquad [\text{I-7a}]$$

where

$$^{\pm}(x-x', z-z_0) = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{+\infty} \exp\left[-j\, q_x\, (x-x')\right] \exp\left[\pm j\, \frac{q_x^2}{2k}\,(z-z_0)\right] d\,q_x \qquad \text{[I-7b]}$$

Completing the square in the exponent by adding and subtracting a term $\frac{j\,k\,(x-x')^2}{2\,(z-z_0)}$

$$^{\pm}(x-x', z-z_0) = \left(\frac{1}{2\pi}\right)\exp\left[\mp\frac{j\,k\,(x-x')^2}{2\,(z-z_0)}\right]\int_{-\infty}^{+\infty}\exp\left(\pm\frac{j\,(z-z_0)}{2\,k}\left\{\left[\frac{k\,(x-x')}{(z-z_0)}\right]^2 \mp \frac{2\,k\,q_x\,(x-x')}{(z-z_0)} + q_x^2\right\}\right) d\,q_x \qquad \text{[I-8]}$$

and using the standard form

$$\int_{-\infty}^{+\infty}\exp\left[-a^2\,u^2\right]\; du = \sqrt{\pi}/a \qquad \text{[I-9]}$$

we obtain

$$A_\alpha^{\pm}(x,y,z,\omega) \approx \pm\left(\frac{1}{j\,\lambda\,(z-z_0)}\right)\exp\left[\mp j\,k\,(z-z_0)\right]\iint A_\alpha^{\pm}(x',y',z_0,\omega)\exp\left\{\mp\frac{j\,k\left[(x-x')^2+(y-y')^2\right]}{2\,(z-z_0)}\right\} d\,x'\,d\,y' \qquad \text{[I-10]}$$

which is the very famous Fresnel diffraction integral or paraxial integral equation. It can be shown that the ***kernel*** or ***Green's function*** of the diffraction integral -- *i.e.*

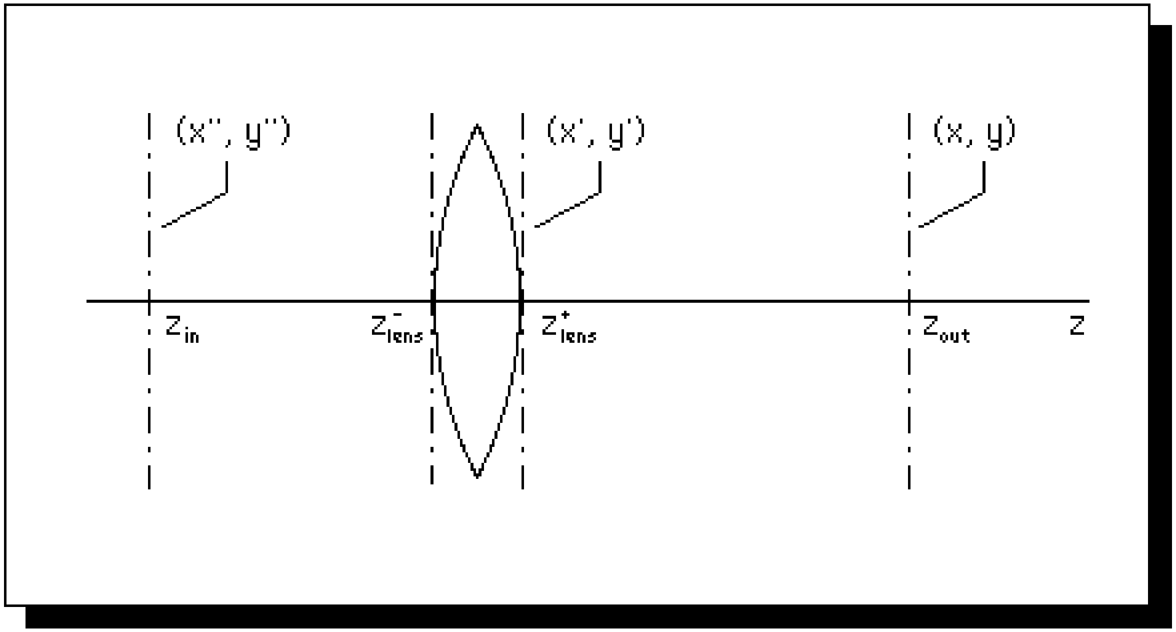
$$h^{\pm}(x-x', y-y'; z-z_0) \equiv \exp\left\{\mp\frac{j\,k\left[(x-x')^2+(y-y')^2\right]}{2\,(z-z_0)}\right\} \qquad \text{[I-11]}$$

-- is a solution of the paraxial wave equation. To recaptitulate, the Fresnel diffraction integral may be expressed

$$A_{\alpha}^{\pm}(x, y, z, \omega) \approx \pm\left(\frac{1}{j\,\lambda\,(z - z_0)}\right)\exp\left[\mp j\,k\,(z - z_0)\right] \iint A_{\alpha}^{\pm}(x', y', z_0, \omega)\, h^{\pm}(x - x', y - y'; z - z_0)\; dx'\, dy' \qquad \text{[I-12]}$$

II. OPTICAL SIGNAL PROCESSING:

In applying this result, let us first consider the simplest unit optical processor.



Optical Signal Processor

We find the field at $z = z_{out}$ in terms of the field at $z = z_{in}$ by successive iterations of the transform in Equation [I-12]. First, make the Fresnel transformation from $z = z_{lens}^{+}$ to $z = z_{out}$[4]

---

[4] In what follows, we assume only fields propagating in the positive z-direction and thus may drop the fussy ± superscript and factors included in earlier more complete expressions.

$$\begin{aligned} \mathrm{A}_\alpha\left(x, y, z_{out}\right) \approx \frac{1}{j\,\lambda\left(z_{out}-z_{lens}\right)} \exp\left[-j\,k\left(z_{out}-z_{lens}\right)\right] \\ \iint \mathrm{A}_\alpha\left(x', y', z_{lens}^{+}\right) \mathrm{h}\left(x-x', y-y'; z_{out}-z_{lens}\right) d\,x'\, d\,y' \end{aligned} \qquad \text{[II-1]}$$

then, make the phase transformation across the thin lens from $z = z_{lens}^{-}$ to $z = z_{lens}^{+}$

$$\begin{aligned} \mathrm{A}_\alpha\left(x, y, z_{out}\right) \approx \frac{1}{j\,\lambda\left(z_{out}-z_{lens}\right)} \exp\left[-j\,k\left(z_{out}-z_{lens}\right)\right] \\ \times \iint \mathrm{A}_\alpha\left(x', y', z_{lens}^{-}\right) \exp\left\{-j\,k\left[n\,D-\frac{\left(x'^{2}+y'^{2}\right)}{2\,f}\right]\right\} \\ \times \mathrm{h}\left(x-x', y-y'; z_{out}-z_{lens}\right) d\,x'\, d\,y' \end{aligned} \qquad \text{[II-2]}$$

and finally, once again make the Fresnel transformation from $z = z_{in}$ to $z = z_{lens}^{-}$

$$\begin{aligned} \mathrm{A}_\alpha\left(x, y, z_{out}\right) \approx \frac{1}{j\,\lambda\left(z_{out}-z_{lens}\right)} \exp\left[-j\,k\left(nD+z_{out}-z_{lens}\right)\right] \left\{\frac{1}{j\,\lambda\left(z_{lens}-z_{in}\right)} \exp\left[-j\,k\left(z_{lens}-z_{in}\right)\right]\right\} \\ \times \iint \left\{\iint \mathrm{A}_\alpha\left(x'', y'', z_{in}\right) \mathrm{h}\left(x'-x'', y'-y''; z_{lens}-z_{in}\right) d\,x''\, d\,y''\right\} \\ \times \exp\left[\frac{j\,k\left(x'^{2}+y'^{2}\right)}{2\,f}\right] \mathrm{h}\left(x-x', y-y'; z_{out}-z_{lens}\right) d\,x'\, d\,y' \end{aligned} \qquad \text{[II-3]}$$

Inverting the order of the integrations and using the definition of kernel in Equation [I-11], we obtain

$$\begin{aligned} \mathrm{A}_\alpha\left(x, y, z_{out}\right) \approx -\frac{\exp\left[-j\,k\left(nD+z_{out}-z_{in}\right)\right]}{\lambda^{2}\left(z_{out}-z_{lens}\right)\left(z_{lens}-z_{in}\right)} \exp\left[-\frac{j\,k\left(x^{2}+y^{2}\right)}{2\left(z_{out}-z_{lens}\right)}\right] \iint \left\{\mathrm{A}_\alpha\left(x'', y'', z_{in}\right) \exp\left[-\frac{j\,k\left(x''^{2}+y''^{2}\right)}{2\left(z_{lens}-z_{in}\right)}\right]\right\} \\ \times \mathbf{I}\left[x'', y'', \left(z_{lens}-z_{in}\right); x, y, \left(z_{out}-z_{lens}\right)\right] d\,x''\, d\,y' \end{aligned} \qquad \text{[II-4]}$$

where

$$\mathbf{I}\left[x'', y'', \left(z_{lens}-z_{in}\right); x, y, \left(z_{out}-z_{lens}\right)\right] \equiv \iint \left\{ \exp\left[-\frac{j\,k}{2}\left(\frac{1}{z_{out}-z_{lens}}+\frac{1}{z_{lens}-z_{in}}-\frac{1}{f}\right)\left(x'^{2}+y'^{2}\right)\right] \times \exp\left[j\,k\,x'\left(\frac{x}{z_{out}-z_{lens}}+\frac{x''}{z_{lens}-z_{in}}\right)\right] \times \exp\left[j\,k\,y'\left(\frac{y}{z_{out}-z_{lens}}+\frac{y''}{z_{lens}-z_{in}}\right)\right]\right\} d\,x'\,d\,y' \qquad \text{[II-5]}$$

If the the thin lens is **infinite in extent** or, equivalently, if we neglect, for the moment, any effect of the finite size of the lens aperature, the result is fairly straight forward and we may write

$$\mathbf{I}\left[x'', y'', \left(z_{lens}-z_{in}\right); x, y, \left(z_{out}-z_{lens}\right)\right] = \mathbf{J}\left(\alpha, \beta_x\right)\,\mathbf{J}\left(\alpha, \beta_y\right) \qquad \text{[I-6]}$$

where

$$\mathbf{J}\left(\alpha, \beta_x\right) \equiv \int_{-\infty}^{+\infty} \exp\left[j\left(\alpha\;x'^{2}+\beta_x\;x'\right)\right] d\,x' \qquad \text{[I-7a]}$$

$$\mathbf{J}\left(\alpha, \beta_y\right) \equiv \int_{-\infty}^{+\infty} \exp\left[j\left(\alpha\;y'^{2}+\beta_y\;y'\right)\right] d\,y' \qquad \text{[I-7b]}$$

$$\alpha \equiv -\frac{k}{2}\left(\frac{1}{z_{out}-z_{lens}}+\frac{1}{z_{lens}-z_{in}}-\frac{1}{f}\right) \qquad \text{[I-7c]}$$

$$\beta_x \equiv k\left(\frac{x}{z_{out}-z_{lens}}+\frac{x''}{z_{lens}-z_{in}}\right) \text{ and } \beta_y \equiv k\left(\frac{y}{z_{out}-z_{lens}}+\frac{y''}{z_{lens}-z_{in}}\right) \qquad \text{[I-7d]}$$

By completing the square in the exponential, we see that

$$\begin{aligned}\mathbf{J}(a,b) &\equiv \int_{-\infty}^{+\infty} \exp\left[j\left(a\,u^2 + b\,u\right)\right] du \\ &= \exp\left[-j\,b^2/4\,a\right] \int_{-\infty}^{+\infty} \exp\left[j\,a\left(u + b/2\,a\right)^2\right] du\end{aligned} \qquad \text{[II-8]}$$

$$\begin{aligned}\mathbf{J}(a,b) &\equiv \int_{-\infty}^{+\infty} \exp\left[j\left(a\,u^2 + b\,u\right)\right] du \\ &= \exp\left[-j\,b^2/4\,a\right] \int_{-\infty}^{+\infty} \exp\left[j\,a\left(u + b/2\,a\right)^2\right] du\end{aligned} \qquad \text{[II-8]}$$

and so the standard form in Equation [I-9] yields

$$\mathbf{J}(a,b) = \begin{cases} \left(j\,\pi/a\right)^{1/2}\ \exp\left[-j\left(b^2/4\,a\right)\right] & \text{for} \quad a \neq 0 \\ 2\,\pi\,\delta(b) & \text{for} \quad a = 0 \end{cases} \qquad \text{[II-9]}$$

Consider, then, two important applications of this result:

1. The field in the image-plane -- *i.e.* $\alpha \equiv 0$, the condition for **imaging** in geometric optics.

$$\begin{aligned}&\mathbf{I}\left[x'', y'', \left(z_{lens} - z_{in}\right); x, y, \left(z_{out} - z_{lens}\right)\right] \\ &\quad = \delta\left(\frac{k\,x}{z_{out} - z_{lens}} + \frac{k\,x''}{z_{lens} - z_{in}}\right)\delta\left(\frac{k\,y}{z_{out} - z_{lens}} + \frac{k\,y''}{z_{lens} - z_{in}}\right)\end{aligned} \qquad \text{[II-10]}$$

and, thus,

$$A_\alpha\left(x, y, z_{out}\right) \approx \left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)\exp\left[-j\,k\left(n\,D+z_{out}-z_{in}\right)\right]\exp\left[-\frac{j\,k\left(x^2+y^2\right)\left(z_{out}-z_{in}\right)}{2\left(z_{out}-z_{lens}\right)^2}\right] \times A_\alpha\left[-\left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)x, -\left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)y, z_{in}\right] \qquad \text{[II-11]}$$

or

$$\left|A_\alpha\left(x, y, z_{out}\right)\right| \approx \left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)\left|A_\alpha\left[-\left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)x, -\left(\frac{z_{lens}-z_{in}}{z_{out}-z_{lens}}\right)y, z_{in}\right]\right| \qquad \text{[II-12]}$$

2. The field in back focal-plane -- *i.e.* $z_{out}-z_{lens}=f$ so that

$$\alpha = -\frac{k}{2\left(z_{lens}-z_{in}\right)}, \ \beta_x \equiv k\left(\frac{x}{f}+\frac{x''}{z_{lens}-z_{in}}\right), \text{ and } \beta_y = k\left(\frac{y}{f}+\frac{y''}{z_{lens}-z_{in}}\right)$$

and, thus,

$$\mathbf{J}\left(\alpha, \beta_x\right) = \left[-\frac{j\,2\pi\left(z_{lens}-z_{in}\right)}{k}\right]^{1/2} \exp\left[j\,\frac{k\left(z_{lens}-z_{in}\right)}{2}\left(\frac{x}{f}+\frac{x''}{z_{lens}-z_{in}}\right)^2\right] \qquad \text{[II-13a]}$$

$$\mathbf{J}\left(\alpha, \beta_y\right) = \left[-\frac{j\,2\pi\left(z_{lens}-z_{in}\right)}{k}\right]^{1/2} \exp\left[j\,\frac{k\left(z_{lens}-z_{in}\right)}{2}\left(\frac{y}{f}+\frac{y''}{z_{lens}-z_{in}}\right)^2\right] \qquad \text{[II-13a]}$$

$$A_\alpha\left(x, y, z_{out}\right) \approx \frac{1}{j\,\lambda\,f}\exp\left[-j\,k\left(n\,D+z_{out}-z_{in}\right)\right]\exp\left[-\frac{j\,k\left(x^2+y^2\right)}{2\,f}\left(1-\frac{z_{lens}-z_{in}}{f}\right)\right] \iint\left\{A_\alpha\left(x'', y'', z_{in}\right)\exp\left[j\,x''\left(k\,\frac{x}{f}\right)\right]\exp\left[j\,y''\left(k\,\frac{y}{f}\right)\right]\right\}d\,x''\,d\,y'' \qquad \text{[II-14]}$$

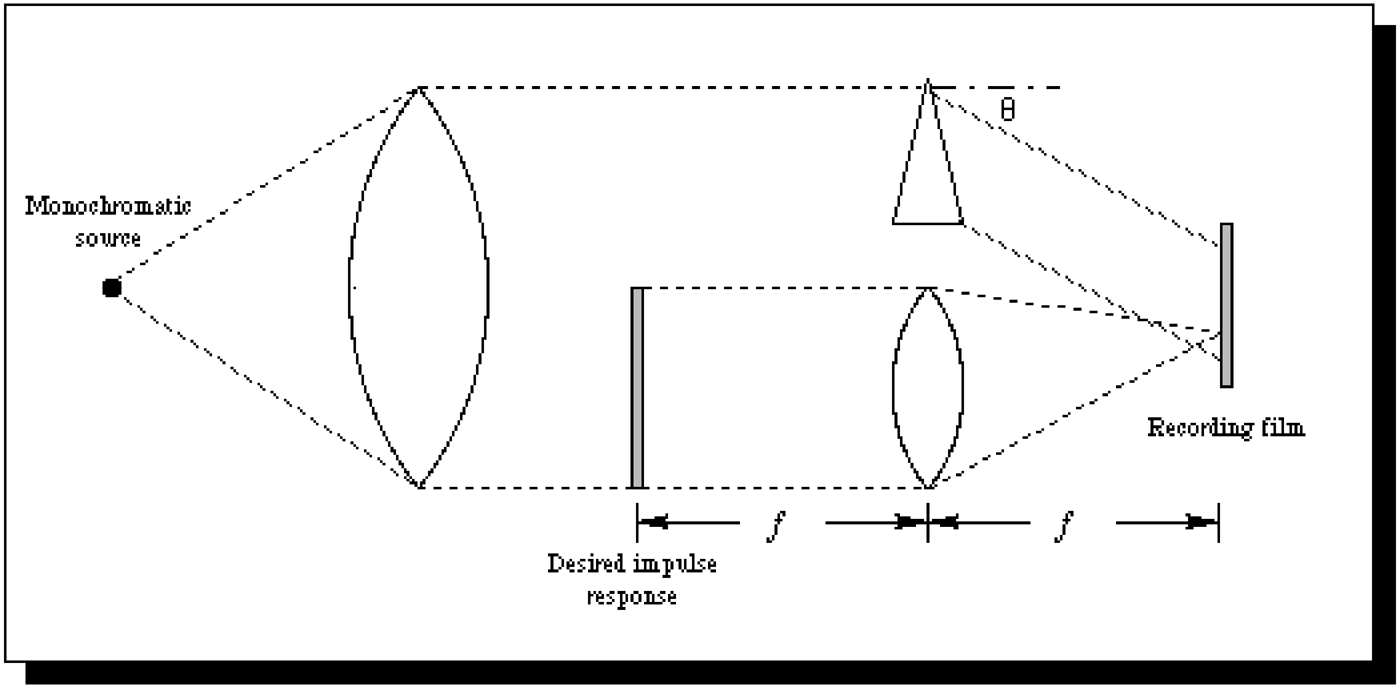
If $$\mathrm{A}_\alpha\left(k_x,k_y\right) \equiv \iint \left\{ \mathrm{A}_\alpha\left(x'',y'',z_{in}\right) \exp\left[j\, x''\, k_x\right] \exp\left[j\, y'\, k_y\right] \right\} d\,x''\, d\,y'' \qquad \text{[II-15]}$$

where $k_x = k\left(x/f\right)$ and $k_y = k\left(y/f\right)$,

$$\mathrm{A}_\alpha\left(x,y,z_{out}\right) \approx \frac{1}{j\,\lambda\, f} \exp\left[-j\,k\left(n\,D + z_{out} - z_{in}\right)\right] \exp\left[-\frac{j\,k\left(x^2+y^2\right)}{2\,f}\left(1-\frac{z_{lens}-z_{in}}{f}\right)\right] \mathrm{A}_\alpha\left(k_x,k_y\right) \qquad \text{[II-16]}$$

SPATIAL FILTERING:

Let us first consider one method for generating **frquency plane masks**.



Modified Mach-Zehnder recording system for generating frequency-masks for a Vander Lugt filter

We suppose have some light distribution $\mathrm{g}\left(x'', y''\right)$ of interest in the front focal-plane of a thin lens, then Equation [II-16] shows that the distribution in the back focal-plane is proportional to $\mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right)$. To "bias" the final processed image, add to this distribution a **reference field** given by $R_0 \exp\left(j\,\alpha\,x\right) \exp\left(-j\,\beta\,z\right)$ where $\alpha = k \sin\theta$ (see figure above). The intensity due to these two fields is then proportional to

$$\left| R_0 \exp\left(j\,\alpha\,x\right) \exp\left(-j\,\beta\,z_{out}\right) + \frac{1}{j\,\lambda\,f} \exp\left[-j\,k\left(n\,D + 2\,f\right)\right] \mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right) \right|^2$$

$$= |R|^2 + \left(\frac{1}{\lambda\,f}\right)^2 \left| \mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right) \right|^2 + \left(\frac{R}{\lambda\,f}\right) \mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right) \exp\left(-j\,\alpha\,x\right) + cc \qquad \text{[II-17]}$$

where all extraneous phase factors are absorbed into R. If we arrange the exposure and developing process so that the amplitude transmittance of the film is proportional to the intensity -- *i.e.* the over all contrast $\gamma \approx 1$ (see the **Hurter-Driffield curve** at the end of this section) that resultant transparency is given by

$$t\left(x, y\right) \quad |R|^2 + \left(\frac{1}{\lambda\,f}\right)^2 \left| \mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right) \right|^2 + \left(\frac{R}{\lambda\,f}\right) \mathrm{G}\left(\frac{k\,x}{f}, \frac{k\,y}{f}\right) \exp\left(-j\,\alpha\,x\right) + cc\ . \qquad \text{[II-18]}$$

If the developed transparency or filter is place in the **frequency-plane** of a classic coherent optical processing system -- *i.e.* to identical thin lenses separated by a distance $2f$ -- and a **transparency to be tested** (distribution $\mathrm{f}\left(x, y\right)$) is placed in the **input-plane** -- *i.e* a distance in front of one lens -- the field distribution just to the right of the frequency-plane, when both transparencies are illuminated with coherent light, is given by

$$A_\alpha\left(x,y,z^{+}_{freq}\right) \quad \frac{|R|^2}{(\lambda f)} F\left(\frac{k x}{f},\frac{k y}{f}\right) + \frac{1}{(\lambda f)^3}\left|G\left(\frac{k x}{f},\frac{k y}{f}\right)\right|^2 F\left(\frac{k x}{f},\frac{k y}{f}\right) + \frac{R}{(\lambda f)^2} G\left(\frac{k x}{f},\frac{k y}{f}\right) F\left(\frac{k x}{f},\frac{k y}{f}\right)\exp\left(-j\alpha x\right) + \frac{R}{(\lambda f)^2} G\left(\frac{k x}{f},\frac{k y}{f}\right) F\left(\frac{k x}{f},\frac{k y}{f}\right)\exp\left(+j\alpha x\right) . \qquad \text{[II-19]}$$

where again extraneous phase factors have been surpressed. The transformation to the output plane of the third and fourth terms yield

$$\mathrm{A}^{(3)}_{\alpha}\left(x,y,z_{out}\right) \approx \frac{R}{\lambda f}\left(\frac{1}{2\pi}\right)^2 \iint \left\{ G\left(k_x,k_y\right) F\left(k_x,k_y\right) \exp\left[j k_x\left(x-\frac{\alpha f}{k}\right)\right]\exp\left[j k_y y\right]\right\} dk_x\, dk_y \qquad \text{[II-20]}$$

$$\mathrm{A}^{(4)}_{\alpha}\left(x,y,z_{out}\right) \approx \frac{R}{\lambda f}\left(\frac{1}{2\pi}\right)^2 \iint \left\{ G\left(k_x,k_y\right) F\left(k_x,k_y\right) \exp\left[j k_x\left(x+\frac{\alpha f}{k}\right)\right]\exp\left[j k_y y\right]\right\} d k_x\, dk_y \qquad \text{[II-21]}$$

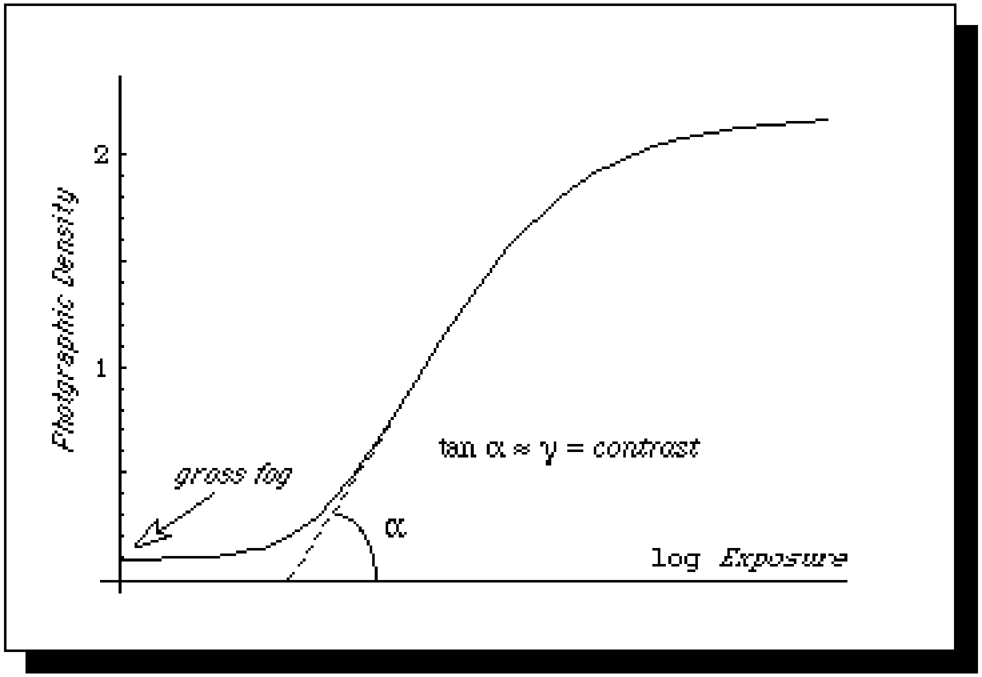
Using the definition in Equation [II-15] we see that

$$\mathrm{A}^{(3)}_{\alpha}\left(x,y,z_{out}\right) \approx \frac{R}{\lambda f} \iint \left\{ \mathrm{g}\left(x',y'\right) \mathrm{f}\left(x-x'-\frac{\alpha f}{k},\, y-y'\right)\right\} d x'\, d y'_1 \qquad \text{[II-22]}$$

is the **convolution** of the two function g and f and

$$\mathrm{A}^{(4)}_{\alpha}\left(x,y,z_{out}\right) \approx \frac{R}{\lambda f} \iint \mathrm{g}\left(x_1,y_1\right) \mathrm{f}\left(x+x_1+\frac{\alpha f}{k},\, y+y_1\right) d x_1\, d y_1 \qquad \text{[II-23]}$$

is their **cross-correlation**.



The Hurter-Driffield Curve (1890)
Characterizes the functional relationship between *photographic density* which is equal to the logarithm of the inverse of the *intensity transmitance* of a developed transparency and *exposure* which is the product of the incident *intensity* and the duration of the exposure.

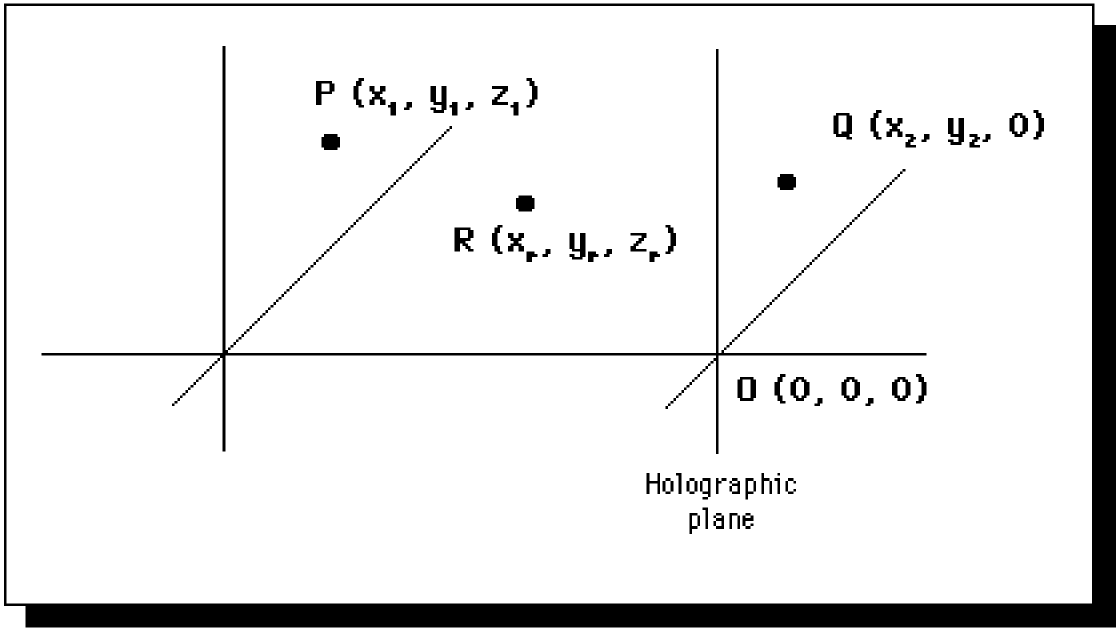
III. IMAGING PROPERTIES OF HOLOGRAPHY:

One of the most important applications of holography is in the construction of beam forming or imaging devices. To study the imaging properties of holograms, we make the following reasonable, but simplifying assumption:

1. The sources of all fields -- *reference, subject, and playback* -- are point sources.

2. Recording medium is thin and has a quadratic response.

3. Fields may be treated in the paraxial approximation.



From the subject point source at P we assume a field

$$\psi_P(\vec{\mathbf{r}},\omega) = C_P \frac{\exp\left[-j\,k_r\,|\vec{\mathbf{r}}-\vec{\mathbf{r}}_P|\right]}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_P|} = |\psi_P|\exp(-j\varphi_P) \qquad \text{[III-1]}$$

and from the reference point source at R we assume a field

$$\psi_R(\vec{\mathbf{r}},\omega) = C_R \frac{\exp\left[-j\,k_r\,|\vec{\mathbf{r}}-\vec{\mathbf{r}}_R|\right]}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}_R|} = |\psi_R|\exp(-j\varphi_R)\ . \qquad \text{[III-2]}$$

The recorded signal in the holographic plane is proportional to

$$I(\vec{\mathbf{r}},\omega) = |\psi_R + \psi_P|^2 = |\psi_R|^2 + |\psi_P|^2 + 2\,|\psi_R||\psi_P|\cos(\varphi_P - \varphi_R) \qquad \text{[III-3]}$$

For convenience,we express the phase of the signals at the point Q taking their phase at O as reference -- *i.e.*

$$\begin{aligned}\varphi_P &= k_r\left(PQ - PO\right)\\ &= k_r\left\{\left[\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + z_1^2\right]^{1/2} - \left[x_1^{\,2} + y_1^{\,2} + z_1^2\right]^{1/2}\right\} \qquad \text{[III-4]}\\ &= k_r\, z_1\left\{\left[1 + \frac{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}{z_1^2}\right]^{1/2} - \left[1 + \frac{x_1^{\,2} + y_1^{\,2}}{z_1^2}\right]^{1/2}\right\}\end{aligned}$$

In the paraxial approximation

$$\varphi_P \approx \frac{k_r}{2\,z_1}\left(x_2^2 + y_2^2 - 2\,x_2\,x_1 - 2\,y_2\,y_1\right) \qquad \text{[III-5]}$$

and similarly

$$\varphi_R \approx \frac{k_r}{2\,z_r}\left(x_2^2 + y_2^2 - 2\,x_2\,x_r - 2\,y_2\,y_r\right)\,. \qquad \text{[III-6]}$$

Therefore, in the paraxial approximation

$$\varphi_P - \varphi_R \approx k_r\left[\left(x_2^2 + y_2^2\right)\left(\frac{1}{2\,z_1} - \frac{1}{2\,z_r}\right) - x_2\left(\frac{x_1}{z_1} - \frac{x_r}{z_r}\right) - y_2\left(\frac{y_1}{z_1} - \frac{y_r}{z_r}\right)\right]. \qquad \text{[III-7]}$$

Examples of types of holograms:

1. **"On Axis" or Gabor Holograms:** ($x_1 = y_1 = x_r = y_r = 0$, $z_1 = -u$ and $z_r = -v$)

$$\varphi_P - \varphi_R \approx \frac{k_r}{2}\left[\left(x_2^2 + y_2^2\right)\left(\frac{1}{v} - \frac{1}{u}\right)\right] = \frac{x_2^2 + y_2^2}{2\,f}\,. \qquad \text{[III-8]}$$

The condition

$$\varphi_P - \varphi_R \approx n\,2\pi \quad \text{or} \quad x_2^2 + y_2^2 = n\,2\,f\,\lambda_r \qquad \text{[III-9]}$$

defines the points on circles of constructive interference or maximum film darkening ("Fresnel" zone plate)

2. **"Off Axis" or Leith-Upatnicks Holograms:** ($x_1 = y_1 = x_r = y_r = 0$, $z_1 = -u$ and $z_r = -v$)

The points of constructive interference or maximum film darkening lie on circles ("off-centered" zone plates) centered at

$$\left(x_2, y_2\right) = \left(\frac{x_r\, z_1 - x_1\, z_r}{z_1 - z_r}, \frac{y_r\, z_1 - y_1\, z_r}{z_1 - z_r}\right) \qquad \text{[III-10a]}$$

with radius

$$\left(\frac{x_r\, z_1 - x_1\, z_r}{z_1 - z_r}\right)^2 + \left(\frac{y_r\, z_1 - y_1\, z_r}{z_1 - z_r}\right)^2 + n\,\lambda_r \left(\frac{2\, z_1\, z_r}{z_1 - z_r}\right) \qquad \text{[III-10b]}$$

RECONSTRUCTION OR PLAYBACK PROCESS

From the playback point source at C we assume a field

$$\psi_{PB}\left(\vec{\mathbf{r}}, \omega\right) = C_{PB}\, \frac{\exp\left[-j\, k_{pb} \left|\vec{\mathbf{r}} - \vec{\mathbf{r}}_{PB}\right|\right]}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}}_{PB}\right|} = \left|\psi_{PB}\right| \exp\left(-j\varphi_{PB}\right) \qquad \text{[III-11]}$$

where we allowing for the posibility of using a different wavelength in the playback process. That process yields, among other things, a **"virtual" signal**

$$\psi_{PB}\,\psi_R\ \ \psi_P = \left|\psi_{PB}\right| \left|\psi_R\right| \left|\psi_P\right| \exp\left(-j\varphi_{real}\right) \qquad \text{[III-12]}$$

where $\varphi_{virtual} = \varphi_{PB} + \varphi_R - \varphi_P$ and a **"real" signal**

$$\psi_{PB}\,\psi_R\;\;\psi_P = |\psi_{PB}|\,|\psi_R|\,|\psi_P| \exp\left(-j\varphi_{real}\right) \qquad \text{[III-13]}$$

where $\varphi_{real} = \varphi_{PB} - \varphi_R + \varphi_P$. The hologram may be photograpnically scaling (enlarged or reduced) by a factor $m \equiv x_2'/x_2 = y_2'/y_2$. If so the phase of the regenerated signals may be expressed as

$$\begin{aligned}\varphi_{\left(\substack{vitual\\ real}\right)} = {} & k_{pb}\left(\frac{1}{2\,z_{pb}}\right)\left(x_2'^{\,2} + y_2'^{\,2} - 2\,x_2'\,x_{pb} - 2\,y_2'\,y_{pb}\right)\\ & \mp k_r\left(\frac{1}{2\,z_r}\right)\left(x_2^2 + y_2^2 - 2\,x_2\,x_r - 2\,y_2\,y_r\right)\\ & \pm k_r\left(\frac{1}{2\,z_1}\right)\left(x_2^2 + y_2^2 - 2\,x_2\,x_1 - 2\,y_2\,y_1\right)\end{aligned} \qquad \text{[III-14a]}$$

or as

$$\begin{aligned}\varphi_{\left(\substack{vitual\\ real}\right)} = {} & k_{pb}\left(\frac{1}{2}\right)\left[\left(x_2'^{\,2} + y_2'^{\,2}\right)\left(\frac{1}{z_{pb}} \pm \frac{\mu}{m^2 z_1} \mp \frac{\mu}{m^2 z_r}\right)\right.\\ & \left. -2\,x_2'\left(\frac{x_{pb}}{z_{pb}} \pm \frac{\mu\,x_1}{m\,z_1} \mp \frac{\mu\,x_r}{m\,z_r}\right) - 2\,y_2'\left(\frac{y_{pb}}{z_{pb}} \pm \frac{\mu\,y_1}{m\,z_1} \mp \frac{\mu\,y_r}{m\,z_r}\right)\right]\end{aligned} \qquad \text{[III-14b]}$$

where $\mu \equiv k_r/k_{pb} = \lambda_{pb}/\lambda_r$. If the holographic process is to "image' a point source, the point image must have over the holographic plane the relative phase

$$\varphi \approx \frac{k_{pb}}{2\,z_3}\left(x_2'^2 + y_2'^2 - 2\,x_2'\,x_3 - 2\,y_2'\,y_3\right) \qquad \text{[III-15]}$$

so that

$$\frac{1}{z_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}}=\frac{1}{z_{pb}}\pm\frac{\mu}{m^2}\frac{1}{z_1}\mp\frac{\mu}{m^2}\frac{1}{z_r} \qquad \text{[III-16a]}$$

$$x_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}=m\left(\frac{m\,z_1\,z_{pb}\,z_r\pm\mu\,z_{pb}\,z_r\,x_1\mp\mu\,z_{pb}\,z_1\,x_r}{m^2\,z_1\,z_r\pm\mu\,z_{pb}\,z_r\mp\mu\,z_{pb}\,z_1}\right) \qquad \text{[III-16b]}$$

$$y_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}=m\left(\frac{m\,z_1\,z_{pb}\,z_r\pm\mu\,z_{pb}\,z_r\,y_1\mp\mu\,z_{pb}\,z_1\,y_r}{m^2\,z_1\,z_r\pm\mu\,z_{pb}\,z_r\mp\mu\,z_{pb}\,z_1}\right). \qquad \text{[III-16c]}$$

and, therefore, the lateral image magnification is given by

$$\frac{\partial}{\partial x_1}x_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}=\frac{\partial}{\partial y_1}y_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}=\pm\frac{\mu}{m}\,\frac{z_{3\begin{pmatrix}\scriptstyle vitual\\ \scriptstyle real\end{pmatrix}}}{z_1} \qquad \text{[III-17]}$$

APPENDIX

THE PARAXIAL WAVE EQUATION

We derive here the paraxial approximation of the Helmholz equation. To that end, we start with the homogeneous Helmholz equation for the electric field in the form

$$\left[\mathrm{div\ grad}\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)+k^{2}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\nabla^{2}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)+k^{2}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[A-1]}$$

where $k^{2}=\omega^{2}\mu\varepsilon(\omega)$. The "searchlight" mode of propagation that we seek is an elaboration of a plane wave propagating in, say, the z-direction, and we assume that a particular component of that field may written in the form

$$\mathbf{E}_{\alpha}(\vec{\mathbf{r}},\omega)=\Psi(\vec{\mathbf{r}},\omega)\exp(-j\,k\,z) \qquad \text{[A-2]}$$

where the function $\Psi(\vec{\mathbf{r}},\omega)$ represents a spatial modulation or "masking" of the plane wave. The z-direction is obviously special and it is, therefore, useful -- nay, essential -- to appropriately parse the spatial differential operators. For the **grad** operator we may write

$$grad\{\text{ anything }\}=\vec{\nabla}\{\text{ anything }\}=\vec{\nabla}_{\mathrm{tr}}\{\text{ anything }\}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\{\text{ anything }\} \qquad \text{[A-3]}$$

where the transverse **grad** operator is given by, for example,

$$\vec{\nabla}_{\mathrm{tr}}\{\text{ anything }\}=\hat{\mathbf{x}}\frac{\partial}{\partial x}\{\text{ anything }\}+\hat{\mathbf{y}}\frac{\partial}{\partial y}\{\text{ anything }\}\ . \qquad \text{[A-4]}$$

Thus, this **grad** operator acting on field component represented in Equation [A-2] may be parsed as

$$\vec{\nabla}\mathbf{E}_{\alpha}(\vec{\mathbf{r}},\omega)=\left\{\vec{\nabla}_{\mathrm{tr}}\Psi(\vec{\mathbf{r}},\omega)+\hat{\mathbf{z}}\left[\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega)-j\,k\,\Psi(\vec{\mathbf{r}},\omega)\right]\right\}\exp(-j\,k\,z) \qquad \text{[A-5]}$$

and the **Laplacian** operator acting on field component represented in Equation [A-2] may be parsed as

$$
\begin{aligned}
{}^{2}\,\mathbf{E}_{\alpha}(\vec{\mathbf{r}},\omega) &\equiv \vec{}\quad \vec{}\;\mathbf{E}_{\alpha}(\vec{\mathbf{r}},\omega) \\
&= \left[\vec{}_{\rm tr} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\left\{\vec{}_{\rm tr}\Psi(\vec{\mathbf{r}},\omega) + \hat{\mathbf{z}}\left[\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega) - j\,k\,\Psi(\vec{\mathbf{r}},\omega)\right]\right\}\exp(-j\,kz) \qquad \text{[A-6]} \\
&= \left[\vec{}_{\rm tr}\quad \vec{}_{\rm tr}\Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^{2}}{\partial z^{2}}\Psi(\vec{\mathbf{r}},\omega) - j\,2k\,\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega) - k^{2}\,\Psi(\vec{\mathbf{r}},\omega)\right]\exp(-j\,kz)
\end{aligned}
$$

where the transverse **Laplacian** operator is given by, for example,

$$ {}^{2}_{\rm tr}\{\text{ anything }\} = \frac{\partial^{2}}{\partial x^{2}}\{\text{ anything }\} + \frac{\partial^{2}}{\partial y^{2}}\{\text{ anything }\} \qquad \text{[A-7]} $$

Therefore, the parsed Helmholz equation (**without approximation**) becomes

$$ {}^{2}_{\rm tr}\,\Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^{2}}{\partial z^{2}}\Psi(\vec{\mathbf{r}},\omega) - j\,2k\,\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega) = 0\ . \qquad \text{[A-8]} $$

The **paraxial approximation** is precisely defined by the condition

$$ \frac{\partial^{2}}{\partial z^{2}}\Psi(\vec{\mathbf{r}},\omega) << 2k\,\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega) \qquad \text{[A-9]} $$

which means that the longitudinal variation in the derivative of the modulation function, $\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega)$, changes very little in a distance comparable to the nominal wavelength of the beam -- *i.e.* $2\pi/k$. In this approximation, we neglect the second term of Equation [A-8] to obtain the equation

$$\boxed{\ {}^{2}_{\text{tr}}\ \Psi(\vec{\mathbf{r}},\omega) - j\ 2k\,\frac{\partial}{\partial z}\Psi(\vec{\mathbf{r}},\omega) \approx 0\ } \qquad \text{[A-10]}$$

which is called the **paraxial approximation** of the wave equation.[5]

---

[5] This is also the form of the famous Schrödinger equation used in quantum mechanics. In the Schrödinger equation the first order derivative is a time derivative (*i.e.* $\{z\}$ $\{t\}$), the Laplacian is a full 3D Laplacian (*i.e.* $\{x, y\}$ $\{x, y, z\}$) and field is a particle field.