

THE PARAXIAL WAVE EQUATION

GAUSSIAN BEAMS IN UNIFORM MEDIA:

In point-to-point communication, we may think of the electromagnetic field as propagating in a kind of "searchlight" mode -- *i.e.* a beam of finite width that propagates in some particular direction. In analyzing this mode of wave propagation, we make use of an important solution to the so call paraxial approximation of the electromagnetic wave equation or, more precisely, the paraxial approximation of the Helmholtz equation.

I. PARAXIAL WAVE EQUATION:

To that end, we first derive the paraxial approximation of the Helmholtz equation and then, in the next section, we examine the free-space *Gaussian Beam* solution(s) of that equation. We start with the homogeneous Helmholtz equation for the electric field in the form

$$[\text{div grad}] \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) + k^2 \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = -\nabla^2 \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) + k^2 \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = 0 \quad [\text{I-1}]$$

where $k^2 = \omega^2 \mu(\omega)$. The "searchlight" mode of propagation that we seek is an elaboration of a plane wave propagating in, say, the z -direction, and we assume that a particular component of that field may written in the form

$$\mathbf{E}(\vec{\mathbf{r}}, t) = \mathbf{E}_0(\vec{\mathbf{r}}) \exp(-j k z) \quad [\text{I-2}]$$

where the function $\mathbf{E}_0(\vec{\mathbf{r}})$ represents a spatial modulation or "masking" of the plane wave. The z -direction is obviously special and it is, therefore, useful -- nay, essential -- to appropriately parse the spatial differential operators. For the **grad** operator we may write

$$\text{grad} \{ \text{anything} \} = \nabla \{ \text{anything} \} = \nabla_{\text{tr}} \{ \text{anything} \} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \{ \text{anything} \} \quad [\text{I-3}]$$

where the transverse **grad** operator is given by, for example,

$$\nabla_{\text{tr}} \{ \text{anything} \} = \hat{\mathbf{x}} \frac{\partial}{\partial x} \{ \text{anything} \} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \{ \text{anything} \} . \quad [\text{I-4}]$$

Thus, this **grad** operator acting on field component represented in Equation [I-2] may be parsed as

$$\vec{\nabla} \mathbf{E}(\vec{\mathbf{r}}, z) = \vec{\nabla}_{\text{tr}}(\vec{\mathbf{r}}, z) + \hat{\mathbf{z}} \frac{\partial}{\partial z}(\vec{\mathbf{r}}, z) - jk(\vec{\mathbf{r}}, z) \exp(-jkz) \quad [\text{I-5}]$$

and the **Laplacian** operator acting on field component represented in Equation [I-2] may be parsed as

$$\begin{aligned} \nabla^2 \mathbf{E}(\vec{\mathbf{r}}, z) &= \nabla_{\text{tr}}^2 \mathbf{E}(\vec{\mathbf{r}}, z) \\ &= \nabla_{\text{tr}}^2 + \hat{\mathbf{z}} \frac{\partial^2}{\partial z^2} \nabla_{\text{tr}}(\vec{\mathbf{r}}, z) + \hat{\mathbf{z}} \frac{\partial}{\partial z}(\vec{\mathbf{r}}, z) - jk(\vec{\mathbf{r}}, z) \exp(-jkz) \quad [\text{I-6}] \\ &= \nabla_{\text{tr}}^2 \nabla_{\text{tr}}(\vec{\mathbf{r}}, z) + \frac{\partial^2}{\partial z^2}(\vec{\mathbf{r}}, z) - j2k \frac{\partial}{\partial z}(\vec{\mathbf{r}}, z) - k^2(\vec{\mathbf{r}}, z) \exp(-jkz) \end{aligned}$$

where the transverse **Laplacian** operator is given by, for example,

$$\nabla_{\text{tr}}^2 \{ \text{anything} \} = \frac{\partial^2}{\partial x^2} \{ \text{anything} \} + \frac{\partial^2}{\partial y^2} \{ \text{anything} \} \quad [\text{I-7}]$$

Therefore, the parsed Helmholtz equation (**without approximation**) becomes

$$\nabla_{\text{tr}}^2(\vec{\mathbf{r}}, z) + \frac{\partial^2}{\partial z^2}(\vec{\mathbf{r}}, z) - j2k \frac{\partial}{\partial z}(\vec{\mathbf{r}}, z) = 0. \quad [\text{I-8}]$$

The **paraxial approximation** is precisely defined by the condition

$$\frac{\partial^2}{\partial z^2}(\vec{\mathbf{r}}, z) \ll 2k \frac{\partial}{\partial z}(\vec{\mathbf{r}}, z) \quad [\text{I-9}]$$

which means that the longitudinal variation in the derivative of the modulation function, $\frac{\partial}{\partial z} (\vec{r},)$, changes very little in a distance comparable to the nominal wavelength of the beam -- *i.e.* $2/k$. In this approximation, we neglect the second term of Equation [I-8] to obtain the equation

$$\nabla_{tr}^2 (\vec{r},) - j 2k \frac{\partial}{\partial z} (\vec{r},) = 0 \tag{I-10}$$

which is called the **paraxial approximation** of the wave equation.¹

II. SOLUTIONS OF THE PARAXIAL WAVE EQUATION²

To inform or motivate our next step, we consider the paraxial approximation of a known solution of the Helmholtz equation -- *viz.* a spherical wave

$$\frac{\exp(-jkr)}{r} = \frac{\exp\left(-jk\sqrt{x^2 + y^2 + z^2}\right)}{\sqrt{x^2 + y^2 + z^2}} = \frac{\exp\left[-jkz\sqrt{1 + (x^2 + y^2)/z^2}\right]}{z\sqrt{1 + (x^2 + y^2)/z^2}} \tag{II-1}$$

$$\frac{\exp(-jkz)}{z} \exp\left[-jk(x^2 + y^2)/2z\right]$$

¹ This is also the form of the famous Schrödinger equation used in quantum mechanics. In the Schrödinger equation the first order derivative is a time derivative (*i.e.* $\{z\} \rightarrow \{t\}$), the Laplacian is a full 3D Laplacian (*i.e.* $\{x, y\} \rightarrow \{x, y, z\}$) and field is a particle field.

² For an alternate treatment of Gaussian Beams see http://www.newport.com/tutorials/Gaussian_Beam_Optics.html

Reflecting on the "quadratic" form of this approximate expression, it is reasonable to look for an axially symmetric solution of the paraxial wave equation in the following form -- *i.e.* a **Gaussian beam**:

$$G(\vec{r}, z) = G(r, z) = A_G \exp(-j P(z)) \exp\left[-\frac{jk^2}{2q(z)} r^2\right] \quad [\text{II-2}]$$

where $r^2 = x^2 + y^2$. We test our conjecture by substituting the Gaussian beam function (*i.e.* Equation [II-2]) into the paraxial wave equation (*i.e.* Equation [I-10]), to wit

$$\exp(-j P(z)) \left[\frac{\partial^2}{\partial r^2} \exp\left[-\frac{jk^2}{2q(z)} r^2\right] - j 2k \frac{\partial}{\partial z} \exp(-j P(z)) \exp\left[-\frac{jk^2}{2q(z)} r^2\right] \right] = 0 \quad [\text{II-3}]$$

Executing the indicated operations, we obtain ³

$$\exp(-j P(z)) \left[\frac{\partial^2}{\partial r^2} \exp\left[-\frac{jk^2}{2q(z)} r^2\right] - \frac{jk^2}{q(z)} r^2 - \frac{jk^2}{q(z)} - j 2k \left(-j \frac{d}{dz} P(z) + \frac{jk^2}{2q^2(z)} \frac{d}{dz} q(z)\right) \right] = 0 \quad [\text{II-4a}]$$

or simplifying

$$\frac{d}{dz} P(z) + \frac{j}{q(z)} + \frac{k^2}{2q^2(z)} \left(1 - \frac{d}{dz} q(z)\right) = 0 \quad [\text{II-4b}]$$

Hence, for an **arbitrary** this equation is separable into two parts - *viz.*

$$\frac{k^2}{2q^2(z)} \left(1 - \frac{d}{dz} q(z)\right) = 0 \quad \text{leads to} \quad \frac{d}{dz} q(z) = 1 \quad [\text{II-4c}]$$

³ Recall that in polar coordinates $\frac{\partial^2}{\partial r^2} f(r) = f''(r) + \frac{1}{r} f'(r)$

and

$$\frac{d}{dz} P(z) + \frac{j}{q(z)} = 0 \quad \text{leads to} \quad \frac{d}{dz} P(z) = \frac{-j}{q(z)} \quad [\text{II-4d}]$$

which are satisfied by the simple solutions

$$q(z) = z + q_0 \quad [\text{II-5a}]$$

$$P(z) = -j \ln[q(z)] = -j \ln[z + q_0] \quad [\text{II-5b}]$$

On comparison with the paraxial approximation of a spherical wave we may write $q(z)$ in terms of a radius of curvature $R(z)$ and a width $w(z)$ -- viz.

$$\frac{1}{q(z)} = \frac{1}{z + q_0} = \frac{1}{R(z)} + \frac{-j 2}{k w^2(z)} \quad [\text{II-6}]$$

To standardize the constants of integration we assume a **plane wavefront** at an arbitrary reference point $z = 0$ -- i.e. we take $R(0) = \infty$. It follows that,

$$\frac{1}{R(0)} = 0 \quad \text{and} \quad \frac{-j 2}{k w^2(0)} = \frac{1}{q_0}$$

or

$$q_0 = j \frac{k w^2(0)}{2} = j \frac{w^2(0)}{L_F} = j L_F \quad [\text{II-7}]$$

where $L_F = k w^2(0)/2 = w^2(0)/\lambda$ is the critical Gaussian beam scaling parameter which is called variously the **Fresnel length**, the **diffraction length**, or the **confocal parameter**. In terms of this parameter Equation [II-6] may be written

$$\frac{1}{q(z)} = \frac{1}{z + q_0} = \frac{1}{R(z)} + \frac{-j 2}{k w^2(z)} = \frac{1}{z + j L_F} = \frac{z - j L_F}{z^2 + L_F^2} \quad [\text{II-8}]$$

Equating real and imaginary parts, we obtain

$$\frac{1}{R(z)} = \frac{z}{z^2 + L_F^2} \quad \text{and} \quad \frac{2}{k w^2(z)} = \frac{L_F}{z^2 + L_F^2} \quad \text{[II-9]}$$

or, finally, in standardized form

$$R(z) = z \left(1 + \frac{L_F^2}{z^2} \right)$$

$$w^2(z) = w^2(0) \left(1 + \frac{z^2}{L_F^2} \right)$$

[II-10]

where $L_F = w^2(0) / \lambda$. Now since Equation [II-5b] may be written

$$\begin{aligned} P(z) &= -j \ln[z + q_0] = -j \ln\left[z + j \frac{L_F}{z}\right] \\ &= -j \ln\left[\sqrt{z^2 + L_F^2}\right] + j \operatorname{atan} \frac{L_F}{z} \end{aligned} \quad \text{[II-11a]}$$

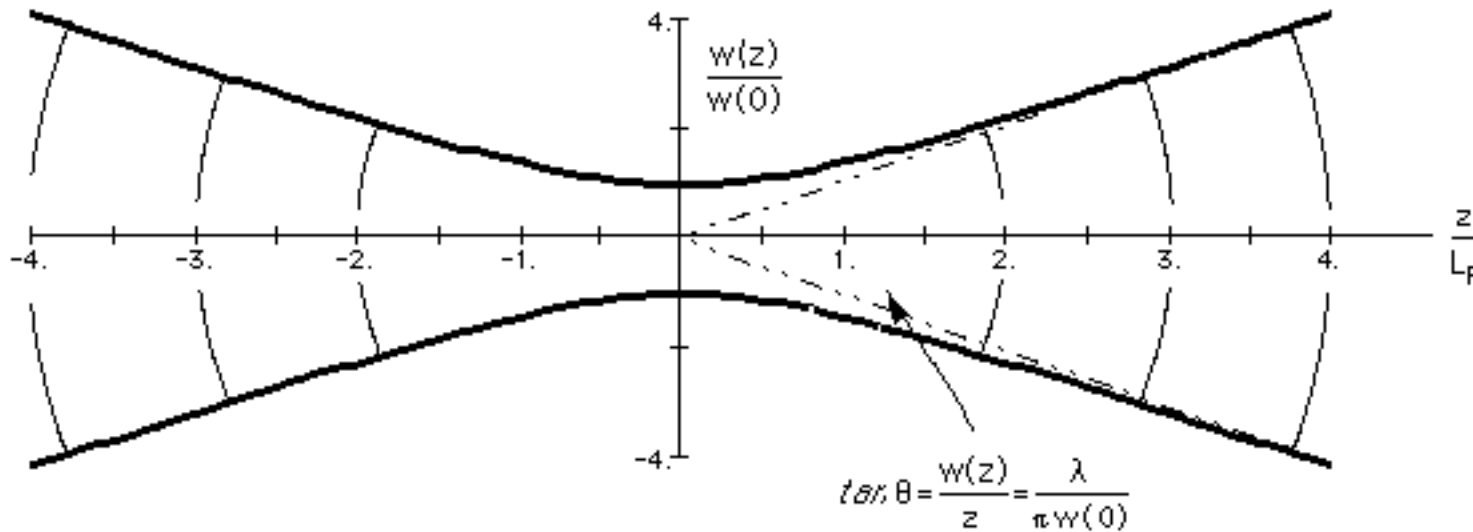
we may write

$$\exp(-j P(z)) = \frac{\exp\left[-j \operatorname{atan} \frac{L_F}{z}\right]}{\sqrt{z^2 + L_F^2}} = \frac{1}{z \sqrt{1 + \frac{L_F^2}{z^2}}} \exp\left[-j \operatorname{atan} \frac{L_F}{z}\right] \quad \text{[II-11b]}$$

to obtain the, **officially approved** complete form of the (lowest order) Gaussian Beam

$$\begin{aligned}
 G(r, z) &= A_G \exp(-j P(z)) \exp -\frac{j k r^2}{2q(z)} \\
 &= A_G \frac{w(0)}{w(z)} \exp -j a \tan \frac{L_F}{z} \exp -\frac{j k r^2}{2R(z)} \exp -\frac{r^2}{w^2(z)}
 \end{aligned}
 \tag{II-12}$$

The following kind of picture is sometimes found to be helpful in understanding the propagation of a Gaussian Beam (the bold curve depicts the spatial variation of the beam width and the light curve the beam curvature at particular points in space):



III. HIGH ORDER HERMITE-GAUSSIAN BEAMS

In order to study the propagation of higher order beams, we substitute the following trial solution:

$$\begin{aligned}
 H_G(x, y, z) &= F(x, y, z) G(x, y, z) \\
 &= f \frac{x}{w} g \frac{y}{w} \exp[-j k (z)] G(x, y, z)
 \end{aligned}
 \tag{III-1}$$

into Equation [I-10], the paraxial wave equation, and obtain

$$\begin{aligned}
 F(x, y, z) \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] G(x, y, z) + 2 \left[\frac{\partial}{\partial x} F(x, y, z) \frac{\partial}{\partial x} G(x, y, z) + \frac{\partial}{\partial y} F(x, y, z) \frac{\partial}{\partial y} G(x, y, z) \right] + G(x, y, z) \frac{\partial^2}{\partial z^2} F(x, y, z) \\
 - 2jk G(x, y, z) \frac{\partial}{\partial z} F(x, y, z) - 2jk F(x, y, z) \frac{\partial}{\partial z} G(x, y, z) = 0
 \end{aligned}
 \tag{III-2}$$

Of course and by design, the first and fifth terms cancel so that the reduced equation becomes

$$\frac{f}{f} + 2ik \frac{dw}{dz} - \frac{w}{q} \left[x \frac{f}{f} + \frac{g}{g} \right] + 2ik \frac{dw}{dz} - \frac{w}{q} \left[y \frac{g}{g} - 2kw^2 \frac{d}{dz} \right] = 0
 \tag{III-3}$$

From Equations [II-8] and [II-9] we see that $\frac{dw}{dz} - \frac{w}{q} = \frac{w}{R} - \frac{w}{R} + \frac{-i2}{kw} = \frac{i2}{kw}$ so that the reduced equation -- *i.e.* Equation [III-3] -- becomes

$$\frac{f}{f} - 4 \left[\frac{f}{f} + \frac{g}{g} \right] - 4 \left[\frac{g}{g} - 2kw^2 \frac{d}{dz} \right] = 0
 \tag{III-4}$$

where $\xi = x/w$ and $\eta = y/w$.

A **Hermite polynomial** of order n^4 has the following differential equation:

⁴ The Hermite polynomials have the generator $H_n(\xi) = (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)$.

$$\frac{d^2}{dz^2} H_n(\xi) - 2 \frac{d}{dz} H_n(\xi) + 2n H_n(\xi) = 0. \quad \text{[III-5]}$$

With the simple change in variables $\xi = \sqrt{2} z / w$ and $\eta = \sqrt{2} y / w$ Equation [III-4] may be written

$$\frac{1}{f} \frac{d^2 f}{dz^2} - 2 \frac{df}{dz} + \frac{1}{g} \frac{d^2 g}{dz^2} - 2 \frac{dg}{dz} - 2kw^2 \frac{d}{dz} = 0 \quad \text{[III-6]}$$

Thus, it is apparent that we can write the functions $f \frac{x}{w}$ and $g \frac{y}{w}$ as Hermite polynomials -- viz.

$$f \frac{x}{w} = H_n(\xi) = H_n \sqrt{2} \frac{x}{w} \quad \text{and} \quad g \frac{y}{w} = H_m(\eta) = H_m \sqrt{2} \frac{y}{w}. \quad \text{[III-7]}$$

if we require that $2kw^2 \frac{d}{dz} = -2(n+m)$. Hence

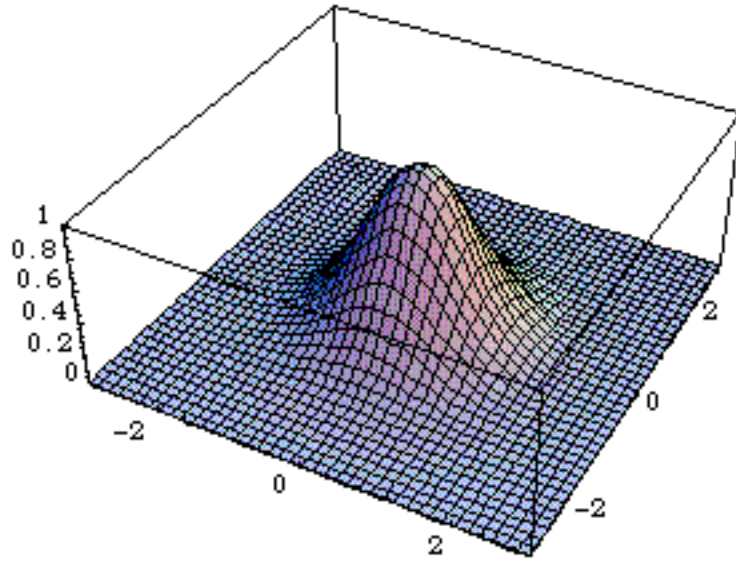
$$\frac{d}{dz} = -\frac{(n+m)}{kw^2} = -\frac{(n+m)L_F}{2[L_F^2 + z^2]} \quad \text{[III-8a]}$$

or $(z) = -(n+m) a \tan(z/L_F). \quad \text{[III-8b]}$

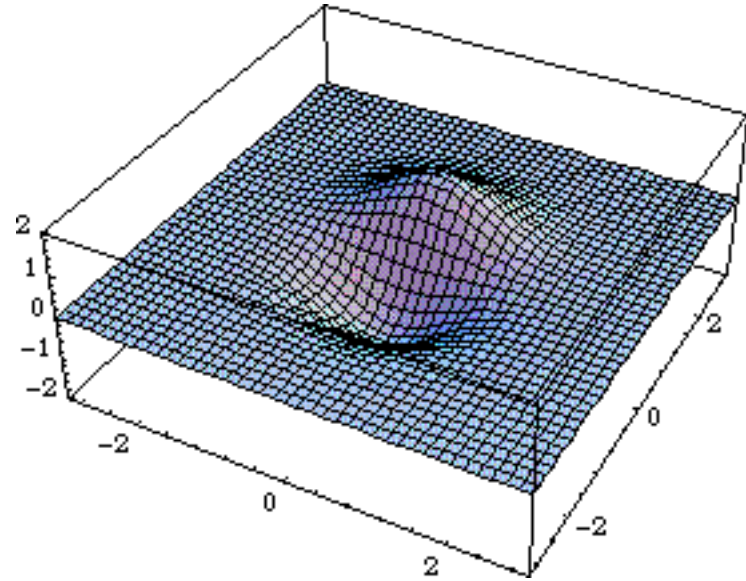
Finally we may write a general solution for the paraxial equation as

$$\begin{aligned} H_{H-G}^{nm}(\xi, \eta, z) = & A_{H-G}^{nm} \frac{w(0)}{w(z)} H_n \sqrt{2} \frac{x}{w} H_m \sqrt{2} \frac{y}{w} \\ & \times \exp[i[n+m+1] \tan^{-1}(z/L_F)] \exp \left[-\frac{ik}{2R(z)} \right] \exp \left[-\frac{z^2}{w^2(z)} \right] \end{aligned} \quad \text{[III-9]}$$

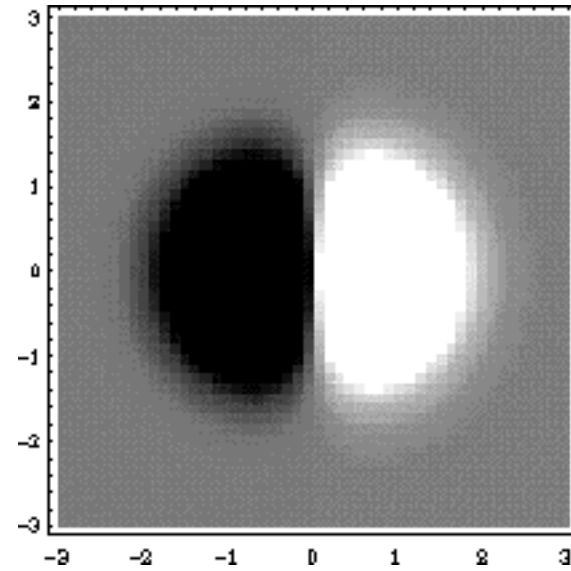
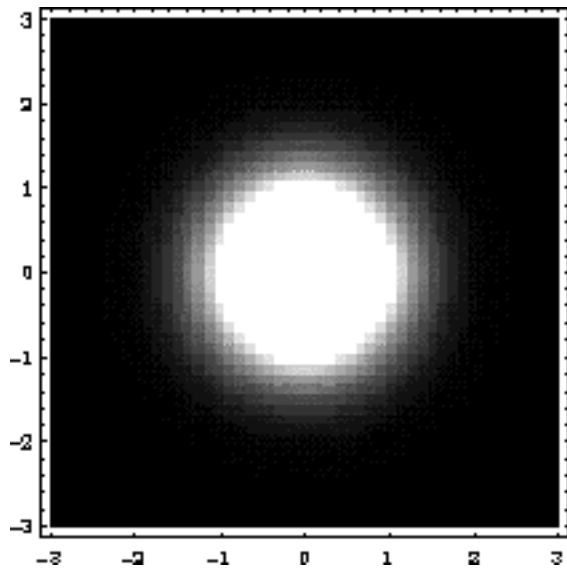
A GALLERY OF HERMITE-GAUSSIAN FIELD DISTRIBUTIONS

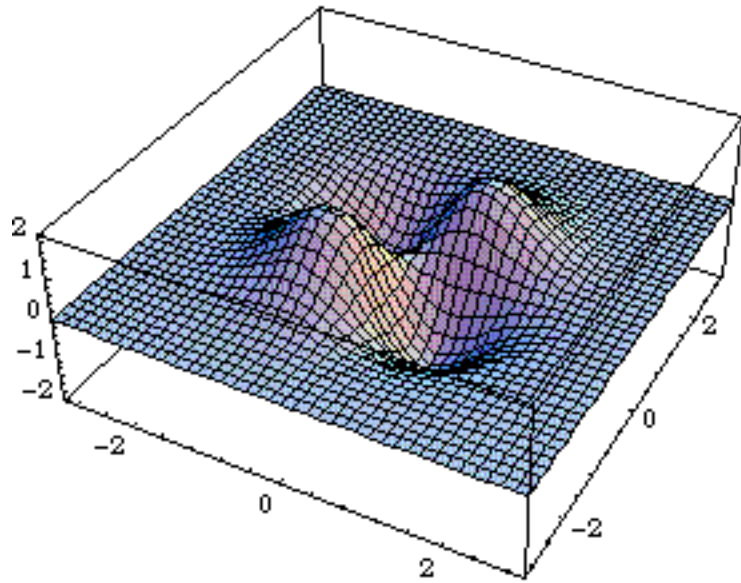


[0, 0] Hermite-Gaussian

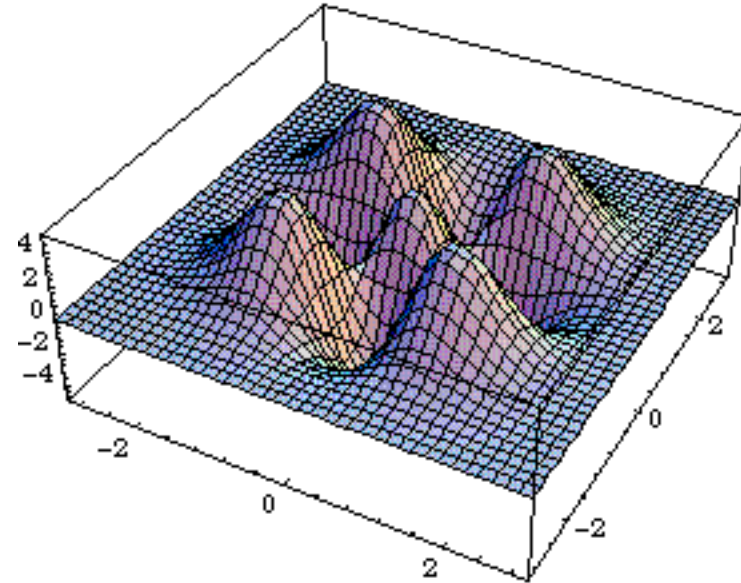


[0, 1] Hermite-Gaussian

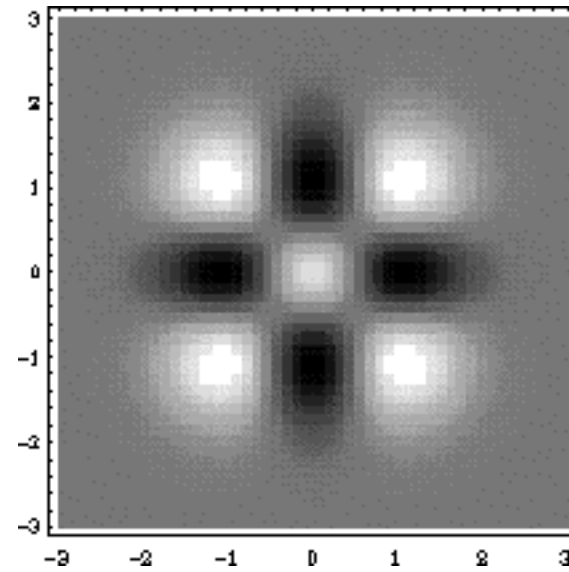
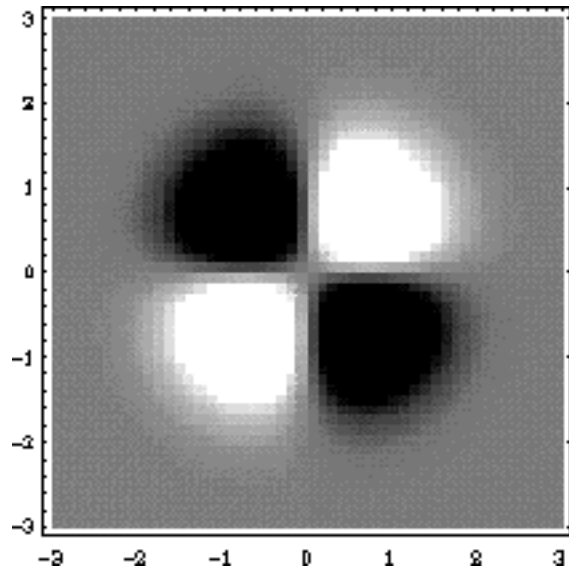




[1, 1] Hermite-Gaussian



[2, 2] Hermite-Gaussian



IV. GAUSSIAN BEAM TRANSFORMATION MATRICES

What we have shown above is that a given Hermite-Gaussian beam is essentially completely specified or defined by the complex function $q(z)$. In propagating through an optical system, the beams are transformed by various optical components. **The amazing fact is that the transformation $q_1 \rightarrow q_2$ produced by a given component follows a simple ABCD transformation law -- viz.**

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \quad \text{[IV-1]}$$

where A, B, C, and D are the matrix elements found in our analysis of geometric optics!!

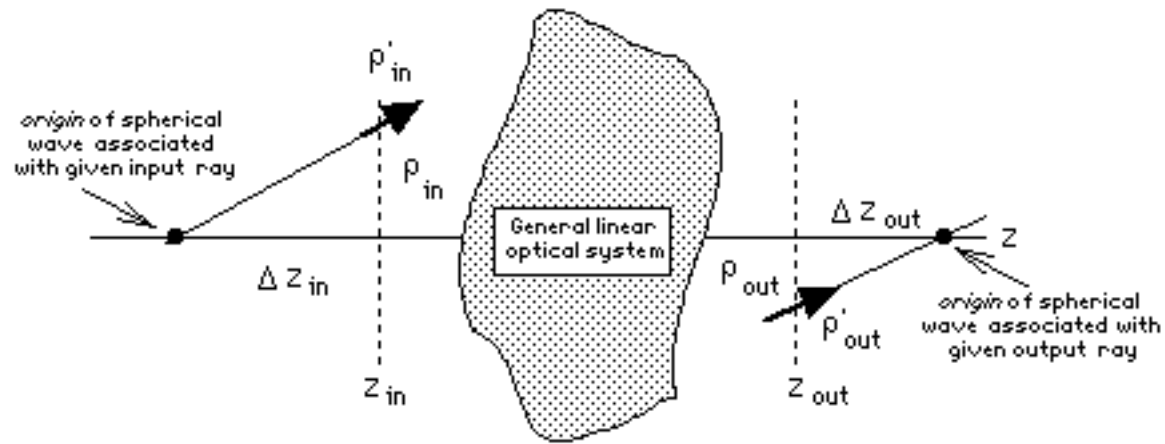
To "**prove**" this, we argue by example. For example, the transformation through a uniform dielectric region is given by

$$q_2 = q_1 + L \quad \text{so that} \quad A = 1 \quad B = L \quad C = 0 \quad D = 1$$

and the transformation through a thin lens is given by

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \quad \text{so that} \quad A = 1 \quad B = 0 \quad C = -\frac{1}{f} \quad D = 1$$

Further "**justification**" may be found in terms of what might be called a " / argument" -- viz. consider the following construction:



From geometric optics and, in particular, Equation [I-11] in our notes entitled *Rays: The Eikonal Treatment of Geometric Optics*, we may write

$$\frac{z_{out}}{\rho_{out}} = A \frac{z_{in}}{\rho_{in}} + B \quad C \frac{z_{in}}{\rho_{in}} + D \quad^{-1} \quad [IV-2a]$$

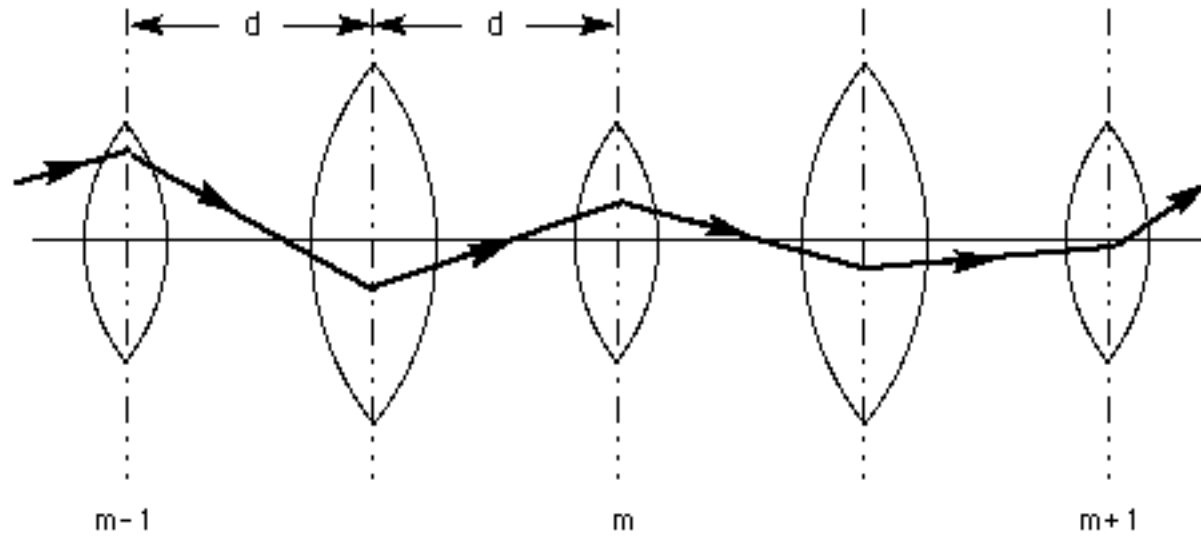
or

$$z_{out} = (A \frac{z_{in}}{\rho_{in}} + B) (C \frac{z_{in}}{\rho_{in}} + D)^{-1} \quad [IV-2b]$$

which is identical to the transformation embodied in Equation [IV-1], if we interpret q as the wave optics generalization of $z = /$.

V. OPTICAL "TRANSMISSION LINES" AND RESONATORS: STABILITY CRITERIA FOR PERIODIC OPTICAL STRUCTURES (INCLUDING RESONATORS) BY RAY OPTIC ANALYSIS

Consider a prototypical periodic guiding lens system (or an equivalent resonator).



Using the **ABCD** matrix, we may write⁵

$$r_{m+1} = A_m r_m + B_m \theta_m \tag{V-1a}$$

$$\theta_{m+1} = C_m r_m + D_m \theta_m \tag{V-1b}$$

From the first equation -- *i.e.* Equation [V-1a] -- we may write

⁵ For the illustrated structure

$$\begin{matrix} A & B \\ C & D \end{matrix} = \begin{matrix} 1 & d & 1 & 0 & 1 & d & 1 & 0 \\ 0 & 1 & -f_2^{-1} & 1 & 0 & 1 & -f_1^{-1} & 1 \end{matrix} = \begin{matrix} (1 - d/f_2)(1 - d/f_1) - d/f_1 & d(2 - d/f_2) \\ d/f_1 f_2 - 1/f_1 - 1/f_2 & 1 - d/f_2 \end{matrix}$$

$$u_m = (u_{m+1} - A u_m) / B \quad \text{and} \quad u_{m+1} = (u_{m+2} - A u_{m+1}) / B$$

and substitute into the second equation -- *i.e.* Equation [V-1b] -- we obtain

$$u_{m+2} - (A + D) u_{m+1} + (AD - BC) u_m = 0 \tag{V-2a}$$

We (or rather you) can show that $[AD - BC] = 1$ so that

$$u_{m+2} - (A + D) u_{m+1} + u_m = 0 . \tag{V-2b}$$

Again we (or you) can also show that

$$\frac{A + D}{2} = 1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2 f_1 f_2} = -1 + 2 \left(1 - \frac{d}{2 f_1} \right) \left(1 - \frac{d}{2 f_2} \right) \tag{V-3}$$

Look for **bounded solutions** of the form $u_m = u_0 \exp(i m \theta)$ which are possible if and only if

$$\exp(i \theta) + \exp(-i \theta) = 2 \cos \theta = A + D = 2 \left(1 - \frac{d}{2 f_1} \right) \left(1 - \frac{d}{2 f_2} \right) \tag{V-4}$$

Therefore propagation is stable -- *i.e.* the rays are confined -- when $\left| \frac{A + D}{2} \right| \leq 1$ so that

$$0 \leq \left(1 - \frac{d}{2 f_1} \right) \left(1 - \frac{d}{2 f_2} \right) \leq 1 . \tag{V-5}$$

Thus, the stability of rays propagation in a periodic system may be usefully characterized in terms of the variables $u_1 = 1 - \frac{d}{2 f_1}$ and $u_2 = 1 - \frac{d}{2 f_2}$ as follows:

STABILITY OR CONFINEMENT DIAGRAM FOR PERIODIC SYSTEMS

