PLANE WAVE SOLUTIONS OF MAXWELL'S EQUATIONS

I. CHARACTERISTICS OF PLANE WAVE SOLUTIONS:

For the record, let us once again restate the form of the macroscopic Maxwell's equations in the time domain which is valid in the high frequency or *optical regime*.

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = -\frac{\partial}{\partial t}\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = -\mu_0\frac{\partial}{\partial t}\vec{\mathbf{H}}(\vec{\mathbf{r}},t) \qquad \text{[I-1a]}$$

$$\vec{\nabla}\times\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \mu_0\,\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},t) = \mu_0\left[\vec{\mathbf{J}}(\vec{\mathbf{r}},t) + \frac{\partial}{\partial t}\vec{\mathbf{P}}(\vec{\mathbf{r}},t)\right] + \varepsilon_0\,\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}(\vec{\mathbf{r}},t) \qquad \text{[I-1b]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \frac{1}{\varepsilon_0}\left[\rho(\vec{\mathbf{r}},t) - \vec{\nabla}\cdot\vec{\mathbf{P}}(\vec{\mathbf{r}},t)\right] \qquad \text{[I-1c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \mu_0\,\vec{\nabla}\cdot\vec{\mathbf{H}}(\vec{\mathbf{r}},t) = 0 \qquad \text{[I-1d]}$$

The most general phenomenological tensorial representation of the linear dielectric response of a given material which incorporates **dissipative** and **anisotropic** effects may be written in the time domain as

$$P_\alpha(\vec{\mathbf{r}},t) = \varepsilon_0 \sum_\beta \int_{-\infty}^{t} dt'\, \chi_{\alpha\beta}(\vec{\mathbf{r}},t-t')\, E_\beta(\vec{\mathbf{r}},t') \qquad \text{[I-2a]}$$

or in the frequency domain as

$$P_\alpha(\vec{\mathbf{r}},\omega) = \varepsilon_0 \sum_\beta \chi_{\alpha\beta}(\vec{\mathbf{r}},\omega)\, E_\beta(\vec{\mathbf{r}},\omega) \qquad \text{[I-2b]}$$

where

$$\chi_{\alpha\beta}(\vec{\mathbf{r}},\omega) = \int_{-\infty}^{t} dt'\, \chi_{\alpha\beta}(\vec{\mathbf{r}},t-t')\exp\left[j\,\omega\,(t-t')\right] = \int_{0}^{\infty} d\tau\, \chi_{\alpha\beta}(\vec{\mathbf{r}},\tau)\exp\left[j\,\omega\,\tau\right] \qquad \text{[I-3]}$$

Neglecting anisotropy, the macroscopic, *optical regime* Maxwell's equations in the frequency domain for a linear, isotropic media may be written

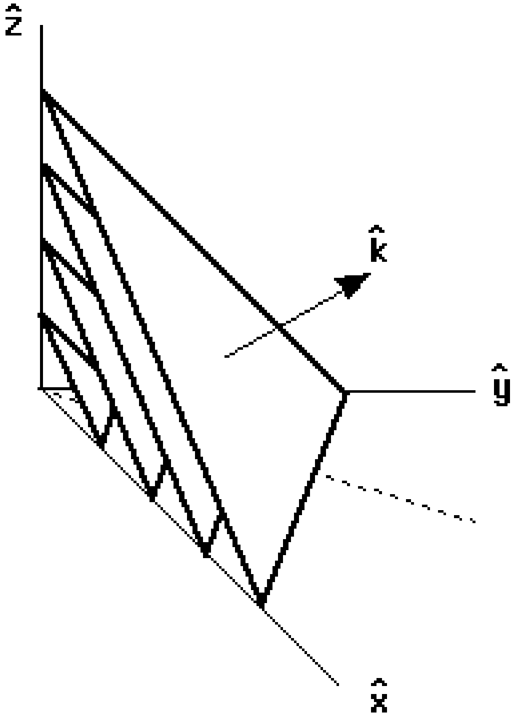
$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\times\varepsilon^{-1}(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega) = -j\,\omega\,\mathbf{B}(\vec{\mathbf{r}},\omega) = -j\omega\,\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-4a]}$$

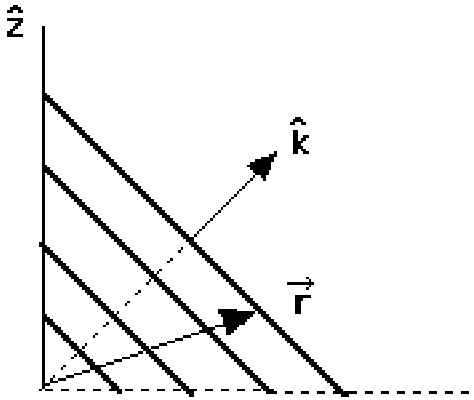
$$\begin{aligned}\vec{\nabla}\times\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega) = \mu_0\,\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) &= \mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega) + j\,\omega\,\mu_0\,\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\\ &= \mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega) + j\,\omega\,\mu_0\,\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)\end{aligned} \qquad \text{[I-4b]}$$

$$\vec{\nabla}\cdot\varepsilon(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\cdot\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega) = \rho(\vec{\mathbf{r}},\omega) \qquad \text{[I-4c]}$$

$$\vec{\nabla}\cdot\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega) = \mu_0\,\vec{\nabla}\cdot\mathbf{H}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-4d]}$$

Consider the possibility of a plane wave solution within a uniform medium -- *i.e.* $\varepsilon(\vec{\mathbf{r}},\omega)\equiv\varepsilon(\omega)$.





of the form

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\,\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\,\exp\left[-j\left(x\,k_x + y\,k_y + z\,k_z\right)\right] \qquad \text{[I-5]}$$

so that the spatial partial derivatives simplify to

$$\frac{\partial}{\partial x}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\,\frac{\partial}{\partial x}\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\left[-j\,k_x\right]\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) \qquad \text{[I-6a]}$$

$$\frac{\partial}{\partial y}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\,\frac{\partial}{\partial y}\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\left[-j\,k_y\right]\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) \qquad \text{[I-6b]}$$

$$\frac{\partial}{\partial z}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\,\frac{\partial}{\partial z}\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\left[-j\,k_z\right]\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right). \qquad \text{[I-6c]}$$

Therefore

$$\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\cdot\vec{\mathbf{E}}(\omega)\,\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = -j\,\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-7a]}$$

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\times\vec{\mathbf{E}}(\omega)\,\exp\left(-j\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = -j\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-7b]}$$

and Maxwell's equations in a **charge/current free region** become

$$-j\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -j\omega\,\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-8a]}$$

$$-j\,\vec{\mathbf{k}}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = j\,\omega\,\varepsilon(\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-8b]}$$

$$-j\,\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-8c]}$$

$$-j\,\vec{\mathbf{k}}\cdot\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[I-8d]}$$

Operate through on both sides of Equation [I-8a] with the operator "$\vec{\mathbf{k}}\times$"

$$\vec{\mathbf{k}} \times \left[\vec{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] = \omega\ \mu_0\ \vec{\mathbf{k}} \times \vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-9a]}$$

and using the "bac-cab" rule[1] and Equation [I-8b] this becomes

$$\vec{\mathbf{k}} \left[\vec{\mathbf{k}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] - \left[\vec{\mathbf{k}} \cdot \vec{\mathbf{k}}\right] \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\omega^2\ \mu_0\ \varepsilon(\omega)\ \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[I-9b]}$$

or finally

$$\left[\vec{\mathbf{k}} \cdot \vec{\mathbf{k}}\right] \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \omega^2\ \mu_0\ \varepsilon(\omega)\ \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad k^2 = \omega^2\ \mu_0\ \varepsilon(\omega) \qquad \text{[I-9c]}$$

Substituting into Equation [I-8a]

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) = (\omega\ \mu_0)^{-1}\ k \left[\hat{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] = \sqrt{\varepsilon(\omega)/\mu_0}\ \left[\hat{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[I-10]}$$

so that the **wave impedance** is given by

$$\eta(\omega) = \left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right| / \left|\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)\right| = \sqrt{\mu_0/\varepsilon(\omega)}\ . \qquad \text{[I-11]}$$

Thus, the complete expression for an electromagnetic plane wave propagating in a direction $\hat{\mathbf{k}}$ is

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \vec{\mathbf{E}}(\omega)\exp\left[-j\left(\vec{\mathbf{r}} \cdot \vec{\mathbf{k}} - \omega\, t\right)\right] \qquad \text{[I-12a]}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}},t) = \left[\eta(\omega)\right]^{-1}\left[\hat{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] \qquad \text{[I-12b]}$$

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[1] That is $\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)$.

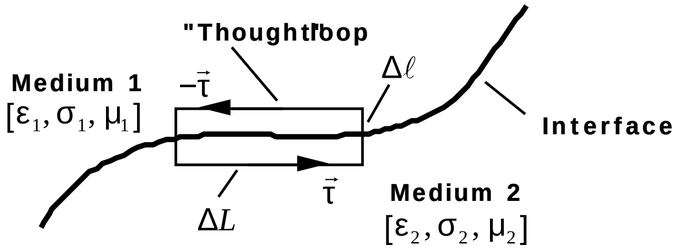
II. ELECTROMAGNETIC INTERFACIAL CONTINUITY CONDITIONS:

The previous section gives a **complete** plane wave solution within a **particular** uniform, linear, isotropic medium. The key remaining problem is to find how that solution may be extended into a second uniform, linear, isotropic medium. The conditions for extending the solution across an interface between two materials are give by consideration of the appropriate integral forms of Maxwell's equations -- *viz*.

$$\oint \vec{\mathbf{E}}(\vec{\mathbf{r}},t)\ \ d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{\mathbf{B}}(\vec{\mathbf{r}},t)\ \ d\vec{A} \qquad \text{[II-1a]}$$

$$\oint \vec{\mathbf{H}}(\vec{\mathbf{r}},t)\ \ d\vec{l} = \iint \vec{\mathbf{J}}(\vec{\mathbf{r}},t)\ \ d\vec{A} + \frac{\partial}{\partial t}\iint \vec{\mathbf{D}}(\vec{\mathbf{r}},t)\ \ d\vec{A} \qquad \text{[II-1b]}$$

Applying these equations to the small ***thought* loop** that spans the interfacial surface, as illustrated below



it is seen that Equation [II-1a] yields

$$\oint \vec{\mathbf{E}}(\vec{\mathbf{r}},t)\ \ d\vec{l} \qquad \left\{\vec{\mathbf{E}}_2(\vec{\mathbf{r}},t) - \vec{\mathbf{E}}_1(\vec{\mathbf{r}},t)\right\}\ \ \vec{\tau}\ \Delta L \qquad 0 \qquad \text{[II-2]}$$

unless $\vec{\mathbf{B}}(\vec{\mathbf{r}},t)$ is **pathologically** large over the loop. Similarly, it is seen that Equation [II-1b] yields

$$\oint \vec{\mathbf{H}}(\vec{\mathbf{r}},t) \quad d\vec{l} \qquad \left\{\vec{\mathbf{H}}_2(\vec{\mathbf{r}},t)-\vec{\mathbf{H}}_1(\vec{\mathbf{r}},t)\right\} \quad \vec{\tau}\ \Delta L \qquad 0 \qquad \text{[II-3]}$$

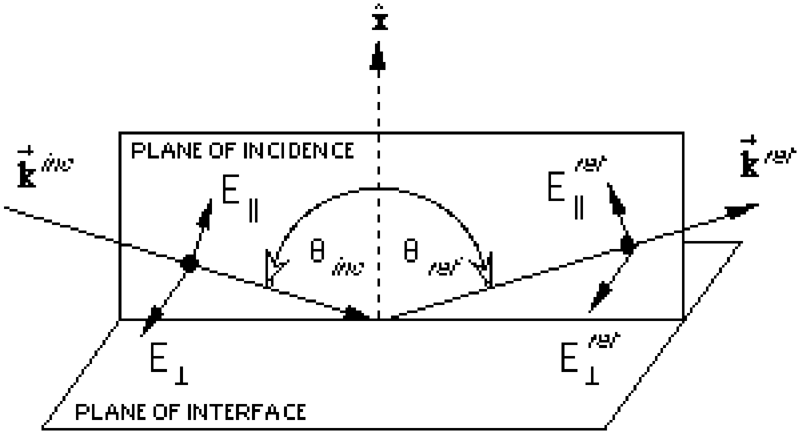
unless $\vec{\mathbf{J}}(\vec{\mathbf{r}},t)$ and/or $\vec{\mathbf{D}}(\vec{\mathbf{r}},t)$ are **pathologically** large over the loop.

> In words and in general, **the tangential component of the electric field strength $\vec{\mathbf{E}}(\vec{\mathbf{r}},t)$ and the magnetic field strength $\vec{\mathbf{H}}(\vec{\mathbf{r}},t)$ are continuous across an interfacial surface between two materials unless electric current density $\vec{\mathbf{J}}(\vec{\mathbf{r}},t)$, the magnetic flux density $\vec{\mathbf{B}}(\vec{\mathbf{r}},t)$, or the electric flux density $\vec{\mathbf{D}}(\vec{\mathbf{r}},t)$ are pathologically large near that interfacial surface.**

III. THE FRESNEL EQUATIONS:

Consider then a plane wave incident on a planar interfacial surface.

The Spatial Configuration:[2]



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[2] Note: In this figure we have taken the *plane of reflection* to be identical to the *plane of incidence*. While assumed here for simplicity, this important identity is establish in the analysis below.

The Mathematical Representation of Fields:

In abstract vector form, the incident field is given by[3]

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{inc}} - \eta_1\,\hat{\mathbf{k}}^{\mathrm{inc}} \times \vec{\mathbf{H}}^{\mathrm{inc}}_{\parallel}\right\}\,\exp\left(-j k_1\,\hat{\mathbf{k}}^{\mathrm{inc}}\;\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{inc}} &= \left\{\vec{\mathbf{H}}^{\mathrm{inc}}_{\parallel} + \eta_1^{-1}\,\hat{\mathbf{k}}^{\mathrm{inc}} \times \vec{\mathbf{E}}^{\,\mathrm{inc}}\right\}\,\exp\left(-j\,k_1\,\hat{\mathbf{k}}^{\mathrm{inc}}\;\vec{\mathbf{r}}\right)\end{aligned} \qquad [\text{III-1a}]$$

the reflected field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{ref}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{ref}} - \eta_1\,\hat{\mathbf{k}}^{\mathrm{ref}} \times \vec{\mathbf{H}}^{\mathrm{ref}}_{\parallel}\right\}\,\exp\left(-j\,k_1\,\hat{\mathbf{k}}^{\mathrm{ref}}\;\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{ref}} &= \left\{\vec{\mathbf{H}}^{\mathrm{ref}}_{\parallel} + \eta_1^{-1}\,\hat{\mathbf{k}}^{\mathrm{ref}} \times \vec{\mathbf{E}}^{\mathrm{ref}}\right\}\,\exp\left(-j\,k_1\,\hat{\mathbf{k}}^{\mathrm{ref}}\;\vec{\mathbf{r}}\right)\end{aligned} \qquad [\text{III-2}]$$

and the transmitted field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{tran}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{tran}} - \eta_2\,\hat{\mathbf{k}}^{\mathrm{tran}} \times \vec{\mathbf{H}}^{\mathrm{tran}}_{\parallel}\right\}\,\exp\left(-j\,k_2\,\hat{\mathbf{k}}^{\mathrm{tran}}\;\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{tran}} &= \left\{\vec{\mathbf{H}}^{\mathrm{tran}}_{\parallel} + \eta_2^{-1}\,\hat{\mathbf{k}}^{\mathrm{tran}} \times \vec{\mathbf{E}}^{\mathrm{tran}}\right\}\,\exp\left(-j\,k_2\,\hat{\mathbf{k}}^{\mathrm{tran}}\;\vec{\mathbf{r}}\right)\end{aligned} \qquad [\text{III-3}]$$

In coordinate form these equations become:

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{\mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}} - \eta_1\left[-\cos\theta_{\mathrm{inc}}\,\hat{\mathbf{x}} + \sin\theta_{\mathrm{inc}}\,\hat{\mathbf{z}}\right] \times \left[\mathrm{H}^{\mathrm{inc}}_{\parallel}\hat{\mathbf{y}}\right]\right\}\,\exp\left[-j\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{inc}} &= \left\{\mathrm{H}^{\mathrm{inc}}_{\parallel}\hat{\mathbf{y}} + \eta_1^{-1}\left[-\cos\theta_{\mathrm{inc}}\,\hat{\mathbf{x}} + \sin\theta_{\mathrm{inc}}\,\hat{\mathbf{z}}\right] \times \left[\mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}}\right]\right\}\,\exp\left[-j\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\end{aligned} \qquad [\text{III-1b}]$$

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{ref}} &= \left\{\mathrm{E}^{\mathrm{ref}}\hat{\mathbf{y}} - \eta_1\left[\cos\theta_{\mathrm{ref}}\,\hat{\mathbf{x}} + \sin\theta_{\mathrm{ref}}\,\hat{\mathbf{z}}\right] \times \left[\mathrm{H}^{\mathrm{ref}}_{\parallel}\hat{\mathbf{y}}\right]\right\}\,\exp\left[-j\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{ref}} &= \left\{\mathrm{H}^{\mathrm{ref}}_{\parallel}\hat{\mathbf{y}} + \eta_1^{-1}\left[\cos\theta_{\mathrm{ref}}\,\hat{\mathbf{x}} + \sin\theta_{\mathrm{ref}}\,\hat{\mathbf{z}}\right] \times \left[\mathrm{E}^{\mathrm{ref}}\hat{\mathbf{y}}\right]\right\}\,\exp\left[-j\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right]\end{aligned} \qquad [\text{III-2b}]$$

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[3] A note on notation: The subscripts and || refer to the polariztion of the electric field taken with respect to the *plane of incidence.* The field components are also called *transverse electric* or TE components and the || field components are called *transverse magnetic* or TM components.

$$\begin{aligned}\vec{\mathbf{E}}^{\rm tran} &= \left\{ \mathrm{E}^{\rm tran}\hat{\mathbf{y}} - \eta_2 \left[-\cos\theta_{\rm tran}\ \hat{\mathbf{x}} + \sin\theta_{\rm tran}\ \hat{\mathbf{z}} \right] \times \left[\mathrm{H}_{\|}^{\rm tran}\hat{\mathbf{y}} \right] \right\} \exp\left[-j\, k_2 \left(-x\cos\theta_{\rm tran} + z\sin\theta_{\rm tran} \right) \right] \\ \vec{\mathbf{H}}^{\rm tran} &= \left\{ \mathrm{H}_{\|}^{\rm tran}\hat{\mathbf{y}} + \eta_2^{-1} \left[-\cos\theta_{\rm tran}\ \hat{\mathbf{x}} + \sin\theta_{\rm tran}\ \hat{\mathbf{z}} \right] \times \left[\mathrm{E}^{\rm tran}\hat{\mathbf{y}} \right] \right\} \exp\left[-j\, k_2 \left(-x\cos\theta_{\rm tran} + z\sin\theta_{\rm tran} \right) \right]\end{aligned} \qquad \text{[III-3b]}$$

Or expanding out the cross-products:

$$\begin{aligned}\vec{\mathbf{E}}^{\rm inc} &= \left\{ \mathrm{E}^{\rm inc}\hat{\mathbf{y}} + \left(\eta_1\, \mathrm{H}_{\|}^{\rm inc} \right) \left[\cos\theta_{\rm inc}\ \hat{\mathbf{z}} + \sin\theta_{\rm inc}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_1 \left(-x\cos\theta_{\rm inc} + z\sin\theta_{\rm inc} \right) \right] \\ \vec{\mathbf{H}}^{\rm inc} &= \left\{ \mathrm{H}_{\|}^{\rm inc}\hat{\mathbf{y}} - \left(\eta_1^{-1}\, \mathrm{E}^{\rm inc} \right) \left[\cos\theta_{\rm inc}\ \hat{\mathbf{z}} + \sin\theta_{\rm inc}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_1 \left(-x\cos\theta_{\rm inc} + z\sin\theta_{\rm inc} \right) \right]\end{aligned} \qquad \text{[III-1c]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\rm ref} &= \left\{ \mathrm{E}^{\rm ref}\hat{\mathbf{y}} + \left(\eta_1\, \mathrm{H}_{\|}^{\rm ref} \right) \left[-\cos\theta_{\rm ref}\ \hat{\mathbf{z}} + \sin\theta_{\rm ref}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_1 \left(x\cos\theta_{\rm ref} + z\sin\theta_{\rm ref} \right) \right] \\ \vec{\mathbf{H}}^{\rm ref} &= \left\{ \mathrm{H}_{\|}^{\rm ref}\hat{\mathbf{y}} - \left(\eta_1^{-1}\, \mathrm{E}^{\rm ref} \right) \left[-\cos\theta_{\rm ref}\ \hat{\mathbf{z}} + \sin\theta_{\rm ref}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_1 \left(x\cos\theta_{\rm ref} + z\sin\theta_{\rm ref} \right) \right]\end{aligned} \qquad \text{[III-2c]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\rm tran} &= \left\{ \mathrm{E}^{\rm tran}\hat{\mathbf{y}} + \left(\eta_2\, \mathrm{H}_{\|}^{\rm tran} \right) \left[\cos\theta_{\rm tran}\ \hat{\mathbf{z}} + \sin\theta_{\rm tran}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_2 \left(-x\cos\theta_{\rm tran} + z\sin\theta_{\rm tran} \right) \right] \\ \vec{\mathbf{H}}^{\rm tran} &= \left\{ \mathrm{H}_{\|}^{\rm tran}\hat{\mathbf{y}} - \left(\eta_2^{-1}\, \mathrm{E}^{\rm tran} \right) \left[\cos\theta_{\rm tran}\ \hat{\mathbf{z}} + \sin\theta_{\rm tran}\ \hat{\mathbf{x}} \right] \right\} \exp\left[-j\, k_2 \left(-x\cos\theta_{\rm tran} + z\sin\theta_{\rm tran} \right) \right]\end{aligned} \qquad \text{[III-3c]}$$

Applying any kind of continuity conditions at the interface requires that

$$\theta_{\rm ref} = \theta_{\rm inc} \qquad \textit{Law of Sinus} \qquad \text{[III-4a]}$$

$$k_2 \sin\theta_{\rm tran} = k_1 \sin\theta_{\rm inc} \qquad \textit{Law of Snell} \qquad \text{[III-4b]}$$

Applying, in particular, the continuity conditions discussed in the previous section -- *viz*.

$$\left[\vec{\mathbf{E}}^{1} \right]_{\rm tang} = \left[\vec{\mathbf{E}}^{2} \right]_{\rm tang} \ \text{ and } \ \left[\vec{\mathbf{H}}^{1} \right]_{\rm tang} = \left[\vec{\mathbf{H}}^{2} \right]_{\rm tang} \qquad \text{[III-5]}$$

at the interface, requires that

$$\begin{aligned}\mathrm{E}^{\rm inc} + \mathrm{E}^{\rm ref} &= \mathrm{E}^{\rm tran} \\ \eta_1^{-1} \cos\theta_{\rm inc} \left[\mathrm{E}^{\rm inc} - \mathrm{E}^{\rm ref} \right] &= \eta_2^{-1} \cos\theta_{\rm tran} \left[\mathrm{E}^{\rm tran} \right]\end{aligned} \qquad \text{[III-6]}$$

and that

$$\begin{gathered} H_{\parallel}^{\text{inc}} + H_{\parallel}^{\text{ref}} = H_{\parallel}^{\text{tran}} \\ \eta_1 \cos\theta_{\text{inc}} \left[H_{\parallel}^{\text{inc}} - H_{\parallel}^{\text{ref}}\right] = \eta_2 \cos\theta_{\text{tran}} \left[H_{\parallel}^{\text{tran}}\right] \end{gathered} \tag{III-7}$$

These two sets of equations yield the Fresnel Reflection Equations

$$\frac{E^{\text{ref}}}{E^{\text{inc}}} = \left[\frac{\eta_1^{-1}\cos\theta_{\text{inc}} - \eta_2^{-1}\cos\theta_{\text{tran}}}{\eta_1^{-1}\cos\theta_{\text{inc}} + \eta_2^{-1}\cos\theta_{\text{tran}}}\right] \tag{III-8a}$$

and

$$\frac{H_{\parallel}^{\text{ref}}}{H_{\parallel}^{\text{inc}}} = \left[\frac{\eta_1 \cos\theta_{\text{inc}} - \eta_2 \cos\theta_{\text{tran}}}{\eta_1 \cos\theta_{\text{inc}} + \eta_2 \cos\theta_{\text{tran}}}\right] \tag{III-9a}$$

Since $\eta_1^{-1} \sin\theta_{\text{inc}} = \eta_2^{-1} \sin\theta_{\text{tran}}$

$$\frac{E^{\text{ref}}}{E^{\text{inc}}} = \frac{\cos\theta_{\text{inc}} \sin\theta_{\text{tran}} - \cos\theta_{\text{tran}} \sin\theta_{\text{inc}}}{\cos\theta_{\text{inc}} \sin\theta_{\text{tran}} + \cos\theta_{\text{tran}} \sin\theta_{\text{inc}}} = \frac{\sin(\theta_{\text{tran}} - \theta_{\text{inc}})}{\sin(\theta_{\text{tran}} + \theta_{\text{inc}})} \tag{III-8b}$$

and

$$\frac{H_{\parallel}^{\text{ref}}}{H_{\parallel}^{\text{inc}}} = \frac{\cos\theta_{\text{inc}} \sin\theta_{\text{inc}} - \cos\theta_{\text{tran}} \sin\theta_{\text{tran}}}{\cos\theta_{\text{inc}} \sin\theta_{\text{inc}} + \cos\theta_{\text{tran}} \sin\theta_{\text{tran}}} = \frac{\tan(\theta_{\text{inc}} - \theta_{\text{tran}})}{\tan(\theta_{\text{inc}} + \theta_{\text{tran}})} \tag{III-9b}$$

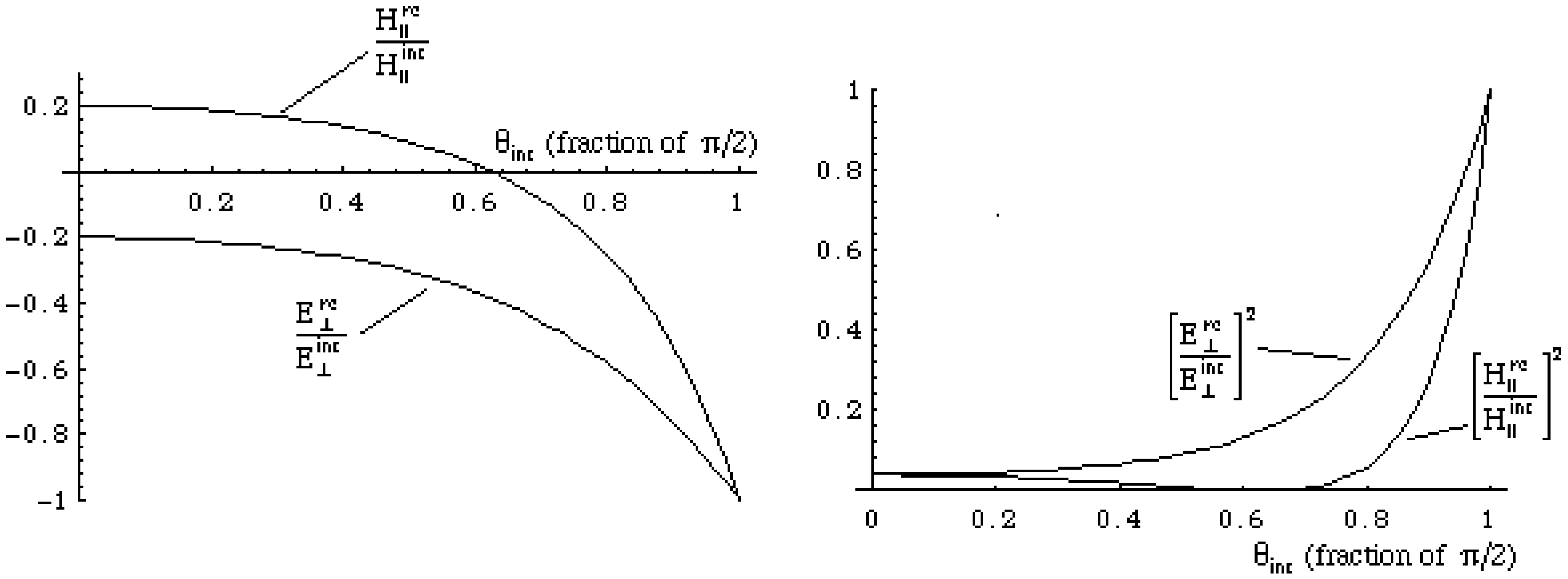
These equations taken together with first equations from Equations [III-6] and [III-7] yield the Fresnel Transmission Equations -- *i.e.*

$$\frac{E^{\text{tran}}}{E^{\text{inc}}} = \frac{2\cos\theta_{\text{inc}} \sin\theta_{\text{tran}}}{\cos\theta_{\text{inc}} \sin\theta_{\text{tran}} + \cos\theta_{\text{tran}} \sin\theta_{\text{inc}}} \tag{III-10}$$

and

$$\frac{H_{\parallel}^{\text{tran}}}{H_{\parallel}^{\text{inc}}} = \frac{2\cos\theta_{\text{inc}} \sin\theta_{\text{inc}}}{\cos\theta_{\text{inc}} \sin\theta_{\text{inc}} + \cos\theta_{\text{tran}} \sin\theta_{\text{tran}}} \tag{III-11}$$

FAMOUS FRESNEL REFLECTION CURVES ($n_2/n_1 = \eta_1/n_2 = \sqrt{\varepsilon_2/\varepsilon_1} = 1.5$)



The minimum (zero) in $H_{\|}^{ref}/H_{\|}^{inc}$ occurs at the Brewster angle where

$$\tan\left(\theta_{inc}^{Brewster} + \theta_{tran}^{Brewster}\right) \quad \infty \qquad \text{[III-12a]}$$

or

$$\theta_{tran}^{Brewster} = \pi/2 - \theta_{inc}^{Brewster} \qquad \text{[III-12b]}$$

or (from Snell's equation)

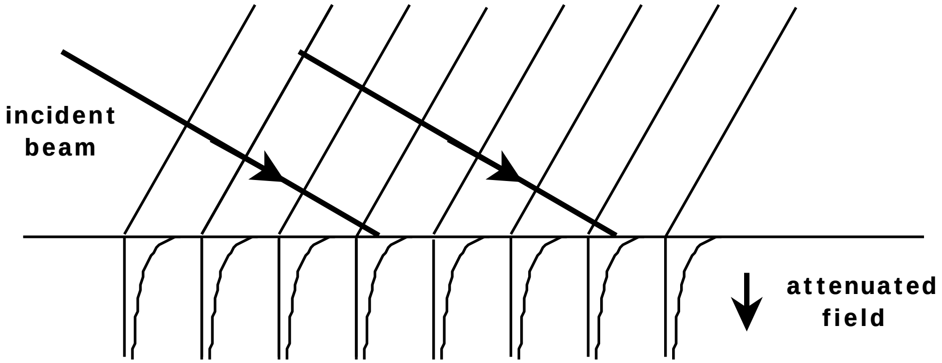
$$\tan\theta_{inc}^{Brewster} = \eta_1/\eta_2 = n_2/n_1 = \sqrt{\varepsilon_2/\varepsilon_1}\ . \qquad \text{[III-12c]}$$

Total Internal Reflection

Reconsider Equation [III-3c] and use Snell's law to write the exponential factors in the form

$$\vec{\mathbf{E}}^{\text{tran}} = \left\{ \mathrm{E}^{\text{tran}}\hat{\mathbf{y}} + \left(\eta_2\, \mathrm{H}_{\|}^{\text{tran}}\right)\left[\cos\theta_{\text{tran}}\,\hat{\mathbf{z}} + \sin\theta_{\text{tran}}\,\hat{\mathbf{x}}\right]\right\} \exp\left[j\,x\sqrt{k_2^2 - k_1^2\sin^2\theta_{\text{inc}}}\right]\exp\left[-j\,z\,k_1\sin\theta_{\text{inc}}\right] \qquad \text{[III-13]}$$

When $\sin\theta_{\text{inc}} > k_2/k_1 = n_2/n_1 \equiv \sin\theta_{\text{inc}}^{\text{crit}}$, $\vec{\mathbf{E}}^{\text{tran}}$, the solution in medium 2, is **attenuated**!



Reconsideration of Equation [III-8a] and [III-9a] shows that the magnitude of the reflection coefficients are **one** when $\sin\theta_{\text{inc}} > \sin\theta_{\text{inc}}^{\text{crit}}$ -- *viz.*

$$\frac{\mathrm{E}^{\text{ref}}}{\mathrm{E}^{\text{inc}}} = \left[\frac{\cos\theta_{\text{inc}} - j\sqrt{\sin^2\theta_{\text{inc}} - \left(k_2/k_1\right)^2}}{\cos\theta_{\text{inc}} + j\sqrt{\sin^2\theta_{\text{inc}} - \left(k_2/k_1\right)^2}}\right] = \exp\left\{-j\,2\tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{\text{inc}} - \left(k_2/k_1\right)^2}}{\cos\theta_{\text{inc}}}\right]\right\} \qquad \text{[III-14a]}$$

and

$$\frac{\mathrm{H}_{\|}^{\text{ref}}}{\mathrm{H}_{\|}^{\text{inc}}} = \left[\frac{\cos\theta_{\text{inc}} - j\sqrt{\left(k_1/k_2\right)^2\sin^2\theta_{\text{inc}} - 1}}{\cos\theta_{\text{inc}} + j\sqrt{\left(k_1/k_2\right)^2\sin^2\theta_{\text{inc}} - 1}}\right] = \exp\left\{-j\,2\tan^{-1}\left[\frac{\sqrt{\left(k_1/k_2\right)^2\sin^2\theta_{\text{inc}} - 1}}{\cos\theta_{\text{inc}}}\right]\right\}. \qquad \text{[III-14b]}$$

IV. A DIFFERENT "TAKE" ON THE FRESNEL EQUATIONS:

Equations [III-1c] through [III-3c] may be rewritten in the following form:

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{ \mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}} + \mathrm{H}_{\|}^{\mathrm{inc}}\left(\eta_1 \cos\theta_{\mathrm{inc}}\right)\hat{\mathbf{z}} + \mathrm{H}_{\|}^{\mathrm{inc}}\left(\eta_1 \sin\theta_{\mathrm{inc}}\right)\hat{\mathbf{x}} \right\} \exp\left[-jk_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right] \\ \vec{\mathbf{H}}^{\,\mathrm{inc}} &= \left\{ \mathrm{H}_{\|}^{\mathrm{inc}}\hat{\mathbf{y}} - \mathrm{E}^{\mathrm{inc}}\left(\eta_1^{-1} \cos\theta_{\mathrm{inc}}\right)\hat{\mathbf{z}} - \mathrm{E}^{\mathrm{inc}}\left(\eta_1^{-1} \sin\theta_{\mathrm{inc}}\right)\hat{\mathbf{x}} \right\} \exp\left[-j\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\end{aligned} \qquad \text{[IV-1a]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{ref}} &= \left\{ \mathrm{E}^{\mathrm{ref}}\hat{\mathbf{y}} - \mathrm{H}_{\|}^{\mathrm{ref}}\left(\eta_1 \cos\theta_{\mathrm{ref}}\right)\hat{\mathbf{z}} + \mathrm{H}_{\|}^{\mathrm{ref}}\left(\eta_1 \sin\theta_{\mathrm{ref}}\right)\hat{\mathbf{x}} \right\} \exp\left[-j\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right] \\ \vec{\mathbf{H}}^{\,\mathrm{ref}} &= \left\{ \mathrm{H}_{\|}^{\mathrm{ref}}\hat{\mathbf{y}} + \mathrm{E}^{\mathrm{ref}}\left(\eta_1^{-1} \cos\theta_{\mathrm{ref}}\right)\hat{\mathbf{z}} - \mathrm{E}^{\mathrm{ref}}\left(\eta_1^{-1} \sin\theta_{\mathrm{ref}}\right)\hat{\mathbf{x}} \right\} \exp\left[-j\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right]\end{aligned} \qquad \text{[IV-1b]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{tran}} &= \left\{ \mathrm{E}^{\mathrm{tran}}\hat{\mathbf{y}} + \mathrm{H}_{\|}^{\mathrm{tran}}\left(\eta_2 \cos\theta_{\mathrm{tran}}\right)\hat{\mathbf{z}} + \mathrm{H}_{\|}^{\mathrm{tran}}\left(\eta_2 \sin\theta_{\mathrm{tran}}\right)\hat{\mathbf{x}} \right\} \exp\left[-j\,k_2\left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}}\right)\right] \\ \vec{\mathbf{H}}^{\,\mathrm{tran}} &= \left\{ \mathrm{H}_{\|}^{\mathrm{tran}}\hat{\mathbf{y}} - \mathrm{E}^{\mathrm{tran}}\left(\eta_2^{-1} \cos\theta_{\mathrm{tran}}\right)\hat{\mathbf{z}} - \mathrm{E}^{\mathrm{tran}}\left(\eta_2^{-1} \sin\theta_{\mathrm{tran}}\right)\hat{\mathbf{x}} \right\} \exp\left[-j\,k_2\left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}}\right)\right]\end{aligned} \qquad \text{[IV-1c]}$$

Notice: If $\boxed{\eta \quad = \eta/\cos\theta}$ **is interpreted as the characteristic impedance of a wave polarized perpendicular to the plane of incidence (TE wave) which is propagating at an angle** θ **with respect to the normal of an interfacial plane and if** $\boxed{\eta_{\|} = \eta\cos\theta}$ **is interpreted as the characteristic impedance of a wave polarized parallel to the plane of incidence (TM wave) which is propagating at an angle** θ **with respect to the normal of an interfacial plane, then the whole analysis of specular reflection fits neatly into transmission line theory.**

In particular, a re-interpretation the famous transmission line equation for the voltage reflection coefficient

$$\Gamma_V(z,\omega) = \frac{Z(z,\omega) - Z_c(\omega)}{Z(z,\omega) + Z_c(\omega)} \qquad \text{[IV-2]}$$

yields an equation for the electric field strength reflection coefficient at the interfacial plane for TE waves

$$\Gamma = \frac{\eta^{\text{tran}} - \eta^{\text{inc}}}{\eta^{\text{tran}} + \eta^{\text{inc}}} = \frac{\eta_2/\cos\theta_{\text{tran}} - \eta_1/\cos\theta_{\text{inc}}}{\eta_2/\cos\theta_{\text{tran}} + \eta_1/\cos\theta_{\text{inc}}} \qquad \text{[IV-3]}$$

which is in precise agreement with Equation [III-8a] -- *i.e.*

$$\frac{E^{\text{ref}}}{E^{\text{inc}}} = \left[\frac{\eta_1^{-1}\cos\theta_{\text{inc}} - \eta_2^{-1}\cos\theta_{\text{tran}}}{\eta_1^{-1}\cos\theta_{\text{inc}} + \eta_2^{-1}\cos\theta_{\text{tran}}}\right].$$

Similarly, a re-interpretation the equally famous transmission line equation for the current reflection coefficient

$$\Gamma_I(z,\omega) = \frac{Y(z,\omega) - Y_c(\omega)}{Y(z,\omega) + Y_c(\omega)} \qquad \text{[IV-4]}$$

yields an equation for the magnetic field strength reflection coeficient at the interfacial plane for TM waves

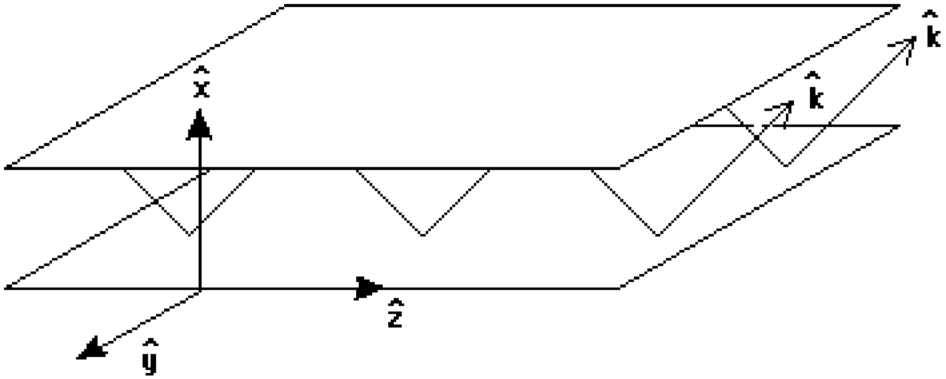
$$\Gamma_{\parallel} = \frac{1/\eta_{\parallel}^{\text{tran}} - 1/\eta_{\parallel}^{\text{inc}}}{1/\eta_{\parallel}^{\text{tran}} + 1/\eta_{\parallel}^{\text{inc}}} = \frac{1/\eta_2\,\cos\theta_{\text{tran}} - 1/\eta_1/\cos\theta_{\text{inc}}}{1/\eta_2\,\cos\theta_{\text{tran}} + 1/\eta_1/\cos\theta_{\text{inc}}} \qquad \text{[IV-5]}$$

which is in precise agreement with Equation [III-9a] -- *i.e.*

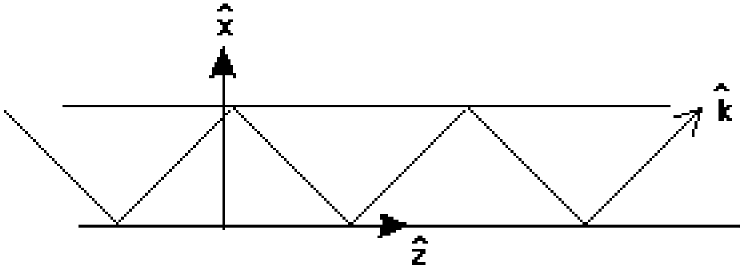
$$\frac{H_{\parallel}^{\text{ref}}}{H_{\parallel}^{\text{inc}}} = \left[\frac{\eta_1\cos\theta_{\text{inc}} - \eta_2\cos\theta_{\text{tran}}}{\eta_1\cos\theta_{\text{inc}} + \eta_2\cos\theta_{\text{tran}}}\right].$$

V. PARALLEL PLATE WAVEGUIDE:

Consider the propagation of a plane wave between two parallel perfectly conducting planes.



Perspective view



Side view

Combining Equations [III-1c] and [III-2c], the electric field strength of the TE wave in the region between the plates may be written

$$\vec{\mathbf{E}} = \hat{\mathbf{y}}\left[\mathrm{E}^{\mathrm{inc}}\exp\left(j\,x\,k_1\cos\theta_{\mathrm{inc}}\right) + \mathrm{E}^{\mathrm{ref}}\exp\left(-j\,x\,k_1\cos\theta_{\mathrm{inc}}\right)\right]\exp\left(-j\,z\,k_1\sin\theta_{\mathrm{inc}}\right) \qquad \text{[V-1]}$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $\mathrm{E}^{\mathrm{ref}} = -\mathrm{E}^{\mathrm{inc}}$ and, therefore,

$$\begin{aligned}\vec{\mathbf{E}} &= \hat{\mathbf{y}}\,\mathrm{E}^{\mathrm{inc}}\left[\exp\left(j\,x\,k_1\cos\theta_{\mathrm{inc}}\right) - \exp\left(-j\,x\,k_1\cos\theta_{\mathrm{inc}}\right)\right]\exp\left(-j\,z\,k_1\sin\theta_{\mathrm{inc}}\right)\\ &= \hat{\mathbf{y}}\,2j\,\mathrm{E}^{\mathrm{inc}}\sin\left(x\,k_1\cos\theta_{\mathrm{inc}}\right)\exp\left(-j\,z\,\beta\right)\end{aligned} \qquad \text{[V-2]}$$

where $\beta = k_1\sin\theta_{\mathrm{inc}}$. At the upper surface -- *i.e.* $x = d$ -- the field parallel to the surface of a perfect conductor must also be zero so that

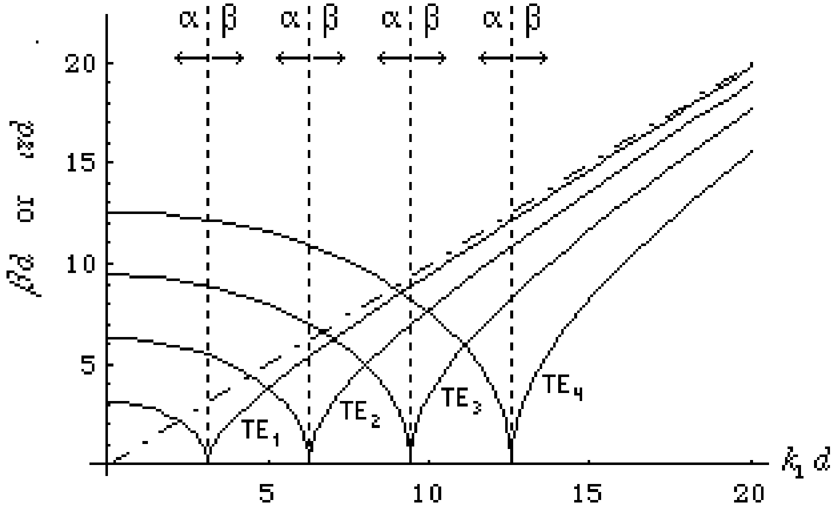
$$d\,k_1\cos\theta_{\mathrm{inc}} = n\,\pi \qquad \text{where } n = 1,2,3,\ldots \qquad \text{[V-3]}$$

and, therefore,

$$\beta_{\mathrm{TE}\,n} = k_1 \sin\theta_{\mathrm{inc}} = \sqrt{k_1^2 - k_1^2 \cos^2\theta_{\mathrm{inc}}} = \sqrt{k_1^2 - \left(n\pi/d\right)^2} \qquad \text{where } n = 1, 2, 3, \ldots \qquad \text{[V-4]}$$

which is **the dispersion relationship for TE waves in a parallel plate waveguide** with "cutoff" frequencies at

$$\omega_n^{cutoff} = \left(n\pi/d\right)\left(\sqrt{\mu_0\,\varepsilon_1}\right)^{-1} \qquad \text{where } n = 1, 2, 3, \ldots \qquad \text{[V-5]}$$



Again combining Equations [III-1c] and [III-2c], the electric field strength of the TM wave in the region between the plates may be written

$$\begin{aligned}\vec{\mathbf{E}}_{\|} = {} & \hat{\mathbf{z}}\left(\eta_1 \cos\theta_{\mathrm{inc}}\right)\left\{\mathrm{H}_{\|}^{\mathrm{inc}} \exp\left(j\,x\,k_1 \cos\theta_{\mathrm{inc}}\right) - \mathrm{H}_{\|}^{\mathrm{ref}} \exp\left(-j\,x\,k_1 \cos\theta_{\mathrm{inc}}\right)\right\} \exp\left(-j\,z\,k_1 \sin\theta_{\mathrm{inc}}\right) \\ & + \hat{\mathbf{x}}\left(\eta_1 \sin\theta_{\mathrm{inc}}\right)\left\{\mathrm{H}_{\|}^{\mathrm{inc}} \exp\left(j\,x\,k_1 \cos\theta_{\mathrm{inc}}\right) + \mathrm{H}_{\|}^{\mathrm{ref}} \exp\left(-j\,x\,k_1 \cos\theta_{\mathrm{inc}}\right)\right\} \exp\left(-j\,z\,k_1 \sin\theta_{\mathrm{inc}}\right)\end{aligned} \qquad \text{[V-6]}$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $\mathrm{H}_{\|}^{\mathrm{ref}} = \mathrm{H}_{\|}^{\mathrm{inc}}$ and, therefore,

$$\begin{aligned}\vec{\mathbf{E}}_{\|} &= \hat{\mathbf{z}}\, 2 j\, \mathrm{H}_{\|}^{\mathrm{inc}} \left(\eta_1 \cos\theta_{\mathrm{inc}}\right) \sin\left(x\, k_1 \cos\theta_{\mathrm{inc}}\right) \exp\left(-j\, z\, \beta\right) \\ &\quad + \hat{\mathbf{x}}\, 2\, \mathrm{H}_{\|}^{\mathrm{inc}} \left(\eta_1 \sin\theta_{\mathrm{inc}}\right) \cos\left(x\, k_1 \cos\theta_{\mathrm{inc}}\right) \exp\left(-j\, z\, \beta\right)\end{aligned} \qquad [\text{V-5}]$$

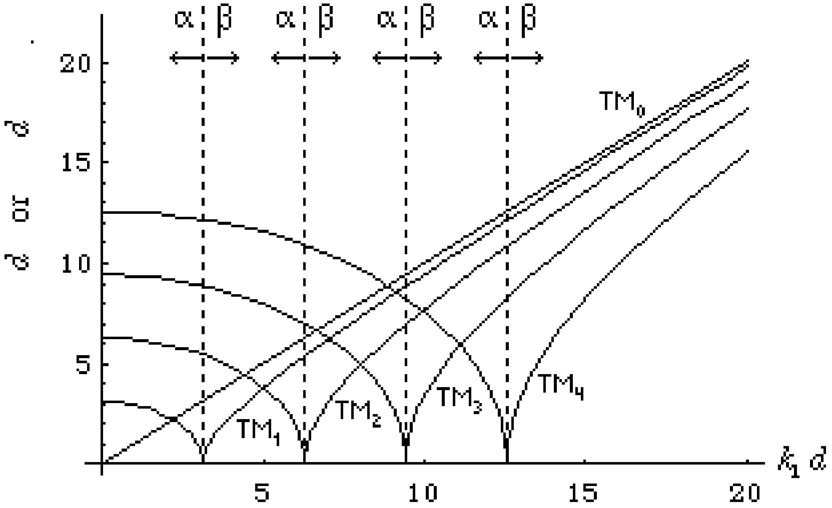
where $\beta = k_1 \sin\theta_{\mathrm{inc}}$. At the upper surface -- *i.e.* $x = d$ -- again the field parallel to the surface of a perfect conductor must also be zero so that

$$d\, k_1 \cos\theta_{\mathrm{inc}} = n\, \pi \qquad \text{where } n = 0, 1, 2, 3, \ldots \qquad [\text{V-7}]$$

and, therefore,

$$\beta_{\mathrm{TM}_n} = k_1 \sin\theta_{\mathrm{inc}} = \sqrt{k_1^2 - k_1^2 \cos^2\theta_{\mathrm{inc}}} = \sqrt{k_1^2 - \left(n\,\pi/d\right)^2} \qquad \text{where } n = 0, 1, 2, 3, \ldots \qquad [\text{V-8}]$$

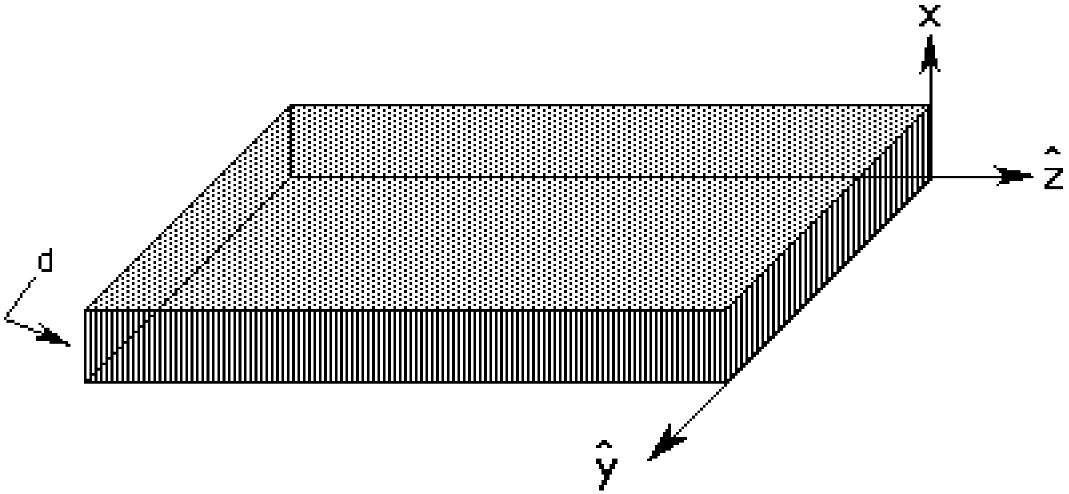
which is **the dispersion relationship for TM waves in a parallel plate waveguide.**



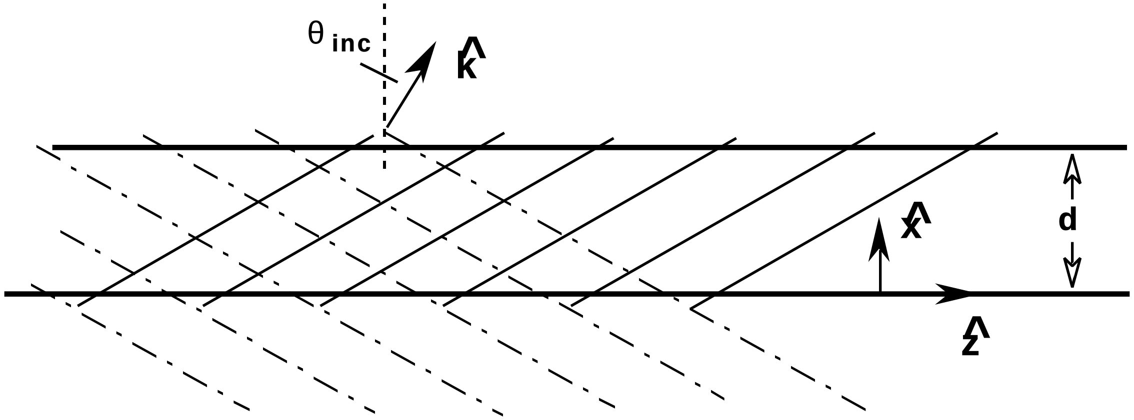
Note that the TM$_0$ mode is a bona fide mode of propagation which does not have a "cutoff" frequency!

VI. DIELECTRIC SLAB WAVEGUIDE:

Consider the propagation of waves "trap in" or "guided by" a dielectric slab of thickness *d*.



In its full generality this is moderately complicated problem, but a rather simple ray optics model of the propagation is sufficient to yield dispersion relationships for the various possible modes of propagation. To obtain such relationships, consider the total internal reflection of a sequence of plane waves as illustrated below.



In order for the multiple reflected wave to be **self-consistence** the following, relatively obvious, phase condition must hold:[4]

$$\Delta\varphi_{x=d} + \Delta\varphi_{x=0} + 2\,k_1\,d\,\cos\theta_{\mathrm{inc}} = m\,2\pi \quad \text{where} \quad m = 0,1,2,3,\ldots \qquad [\mathrm{VI}\text{-}1]$$

where $\Delta\varphi_{x=d}$ and $\Delta\varphi_{x=0}$ are, respectively, the phase shifts associated with the reflections at the upper and lower dielectric boundaries.

For **TE-modes of propagation** Equation [III-14a] gives the phase shift at the boundary (called in the trade *the TE Goos-Hänchen shift*) and Equation [VI-1] becomes

$$-2\tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{\mathrm{inc}} - \sin^2\theta_{\mathrm{inc}}^{\mathrm{crit}}}}{\cos\theta_{\mathrm{inc}}}\right] + k_1\,d\,\cos\theta_{\mathrm{inc}} = m\,\pi \qquad [\mathrm{VI}\text{-}2]$$

where $\sin\theta_{\mathrm{inc}}^{\mathrm{crit}} \equiv k_2/k_1 = n_2/n_1 = \sqrt{\varepsilon_2/\varepsilon_1}$. Therefore, the **self-consistence relationship for TE** is given by

$$\frac{\sqrt{\sin^2\theta_{\mathrm{inc}} - \sin^2\theta_{\mathrm{inc}}^{\mathrm{crit}}}}{\cos\theta_{\mathrm{inc}}} = \tan\left[\frac{n_1\,k_0\,d\,\cos\theta_{\mathrm{inc}}}{2} - m\,\frac{\pi}{2}\right] \qquad [\mathrm{VI}\text{-}3]$$

where $k_0 = \omega/c = 2\pi/\lambda_0$. This is a transcendental equation in the single variable $\cos\theta_{\mathrm{inc}}$. Its solutions yield the allowed bounce angles, $\left(\theta_{\mathrm{inc}}\right)_m$, of possible modes and, hence, the allowed propagation constants since $\beta = k_1 \sin\theta_{\mathrm{inc}}$ The left and right sides of this equation may be plot as a function of $\cos\theta_{\mathrm{inc}}$ with $n_1\,k_0\,d = n_1\,2\pi\left(d/\lambda_0\right)$ and $\sin\theta_{\mathrm{inc}}^{\mathrm{crit}} = n_2/n_1$ as a parameters. The intersections of such curves yield the allowed bounce angles as illustrated below

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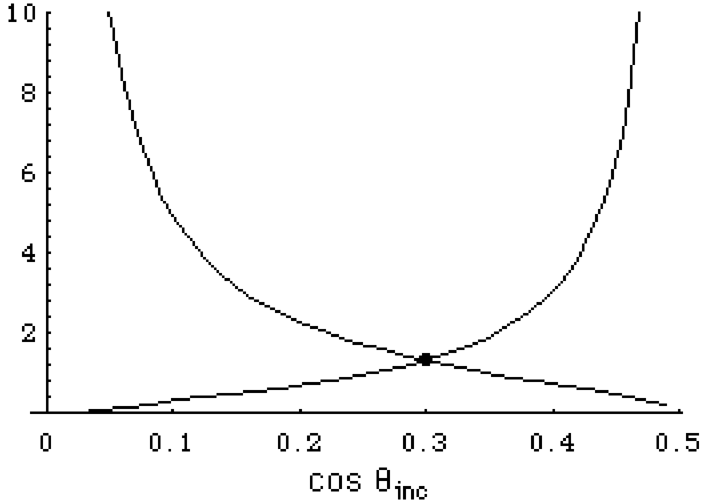
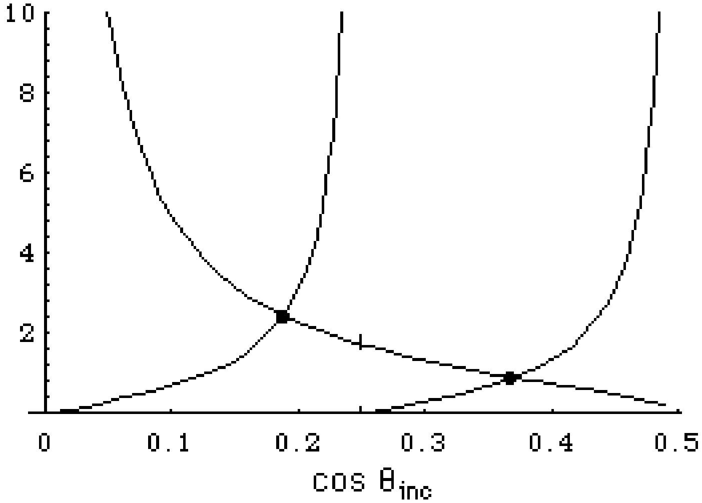
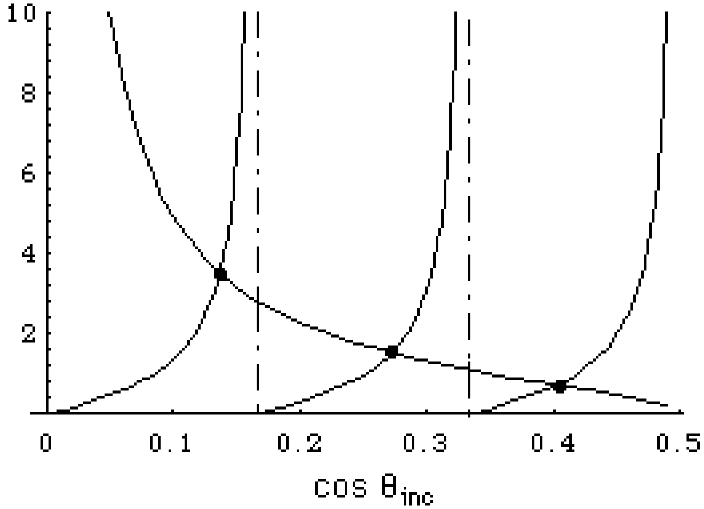
[4] This equation is a direct generalization of Equations [V-3] and [V-7] which figure in our analysis of parallel plane waveguides.



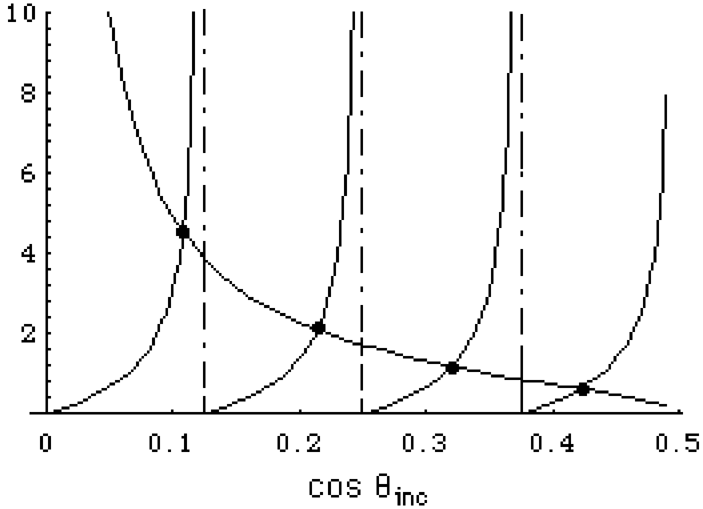
LHS and RHS of Equation [VI-3]
for $n_1 \left(d/\lambda_0\right) = 0.5$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$



LHS and RHS of Equation [VI-3]
for $n_1 \left(d/\lambda_0\right) = 1.0$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$



LHS and RHS of Equation [VI-3]
for $n_1 \left(d/\lambda_0\right) = 1.5$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$



LHS and RHS of Equation [VI-3]
for $n_1 \left(d/\lambda_0\right) = 2.0$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$