

ELECTROMAGNETIC RADIATION -- THE BASICS

A RESTATEMENT OF MACROSCOPIC MAXWELL'S EQUATIONS IN THE FREQUENCY DOMAIN -- VALID FOR LINEAR, LOCAL, ANISOTROPIC MEDIA IN THE *OPTICAL REGIME*.

$$\nabla \times \vec{E}(\vec{r}, t) = -\dot{\vec{B}}(\vec{r}, t) \quad [1a]$$

$$\begin{aligned} \nabla \times \vec{B}(\vec{r}, t) &= \mu_0 \nabla \times \vec{H}(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t) + i \mu_0 \omega \vec{E}(\vec{r}, t) \\ &= \mu_0 \vec{J}(\vec{r}, t) + i \mu_0 \vec{D}(\vec{r}, t) \end{aligned} \quad [1b]$$

$$\nabla \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0} \quad [1c]$$

$$\nabla \cdot \vec{B}(\vec{r}, t) = 0 \quad [1d]$$

HELMHOLTZ EQUATIONS FOR THE FREQUENCY DOMAIN VECTOR AND SCALAR POTENTIALS IN A UNIFORM, LINEAR, ISOTROPIC DIELECTRIC

First, we define the (Magnetic) Vector Potential as

$$\vec{H}(\vec{r}, t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}, t) \quad [2]$$

which, by design, automatically satisfies one of Maxwell's equation -- viz. [1d] -- since

$$\text{div curl} \{ \text{anything} \} = \nabla \cdot (\nabla \times \{ \text{anything} \}) = 0 \quad [3]$$

Next, we introduce the (Electric) Scalar Potential in the form

$$\vec{E}(\vec{r}, t) = -\dot{\vec{A}}(\vec{r}, t) - \nabla \phi(\vec{r}, t) \quad [4]$$

which, by design, automatically satisfies another Maxwell equation -- viz. [1a] -- since

$$\text{curl grad} \{ \text{anything} \} = \nabla \times \nabla \{ \text{anything} \} = 0 \quad [5]$$

Therefore, for **uniform, isotropic media** of Equation [1b] becomes ¹

$$\begin{aligned} \nabla \times \nabla \times \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) &= \mu_0 \vec{\mathbf{J}}(\vec{\mathbf{r}}, t) + \nabla^2 \mu_0 (\) \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) - i \mu_0 (\) \dot{\vec{\mathbf{r}}}(t) \\ &= \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) - \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) \end{aligned} \quad [6a]$$

and Equation [1c] becomes

$$-i \mu_0 (\) \dot{\vec{\mathbf{r}}}(t) - \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t). \quad [6b]$$

$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t)$ is as yet undefined. For our purposes, the most convenient and/or revealing development is in terms of the so called **Lorentz gauge** in which

$$\nabla \cdot \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = -i \mu_0 (\) \dot{\vec{\mathbf{r}}}(t) \quad [7]$$

-- to simplify Equations [6a] and [6b]. Thus

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) + \nabla^2 \mu_0 (\) \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = -\mu_0 \vec{\mathbf{J}}(\vec{\mathbf{r}}, t) \quad [6a]$$

and

$$\nabla^2 \vec{\mathbf{r}}(t) + \nabla^2 \mu_0 (\) \vec{\mathbf{r}}(t) = -\frac{1}{(\)} \dot{\vec{\mathbf{r}}}(t) \quad [6b]$$

Therefore, **both** $\vec{\mathbf{A}}(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{r}}(t)$ **satisfy inhomogeneous** (and homogeneous) **Helmholz equations!**

¹ Here we use the vector identity

$$\nabla \times (\nabla \times \vec{\mathbf{V}}) = \nabla (\nabla \cdot \vec{\mathbf{V}}) - (\nabla^2 \vec{\mathbf{V}})$$

which follows from

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}).$$

APPENDIX: ON SPHERICAL WAVES

We need to establish that spherical waves are valid solutions of Maxwell's equation (or, more precisely, of the inhomogeneous Helmholtz equation derived from Maxwell's equations)

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) + k^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = -\mu \vec{\mathbf{J}}(\vec{\mathbf{r}}, t) \quad [\text{A-1}]$$

1. Let us first look for solutions in "current-free" region so that

$$\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) + k^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = 0 \quad [\text{A-2}]$$

Our goal is to find a solution which depends only on the magnitude of the observer's position vector and, thus, we look for solutions in the form

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{a}} \frac{f(r, t)}{r} \quad [\text{A-3}]$$

where $r = |\vec{\mathbf{r}}|$ and $\vec{\mathbf{a}}$ is a constant vector. To see if this form of solution works, we need to find $\text{div grad } \frac{f(r, t)}{r}$. To that end, we first use the "chain rule" to find $\text{grad } \frac{f(r, t)}{r}$

-- viz.

$$\begin{aligned} \text{grad } \frac{f(r, t)}{r} &= - \frac{f(r, t)}{r^2} \vec{\mathbf{r}} + \left[\frac{\partial}{\partial r} \right] \frac{f(r, t)}{r} \vec{\mathbf{r}} \\ &= \left[\frac{\partial}{\partial r} \right] \frac{1}{r} \frac{d}{dr} f(r, t) - \frac{1}{r^2} f(r, t) \vec{\mathbf{r}} \end{aligned} \quad [\text{A-4}]$$

However

$$\begin{aligned}
 \vec{r} &= \left[\sqrt{x^2 + y^2 + z^2} \right] \\
 &= \frac{x \hat{x}}{\sqrt{x^2 + y^2 + z^2}} + \frac{y \hat{y}}{\sqrt{x^2 + y^2 + z^2}} + \frac{z \hat{z}}{\sqrt{x^2 + y^2 + z^2}} \\
 &= \frac{\vec{r}}{r} = \hat{r}
 \end{aligned} \tag{A-5}$$

so that

$$\begin{aligned}
 \nabla^2 \vec{A}(\vec{r}, t) &= \nabla^2 \vec{A}(\vec{r}, t) = \nabla^2 \vec{a} \frac{f(r, t)}{r} \\
 &= \vec{a} \nabla^2 \frac{f(r, t)}{r} \\
 &= \vec{a} \hat{r} \frac{1}{r} \frac{d}{dr} f(r, t) - \frac{1}{r^2} f(r, t)
 \end{aligned} \tag{A-6}$$

Again, using the "chain rule"

$$\begin{aligned}
 \nabla^2 \vec{A}(\vec{r}, t) &= \vec{a} \hat{r} \frac{1}{r} \frac{d}{dr} f(r, t) - \frac{1}{r^2} f(r, t) \\
 &= \vec{a} \left[\frac{d}{dr} \frac{1}{r} \frac{d}{dr} f(r, t) - \frac{1}{r^3} f(r, t) \right. \\
 &\quad \left. + 3 \frac{1}{r^2} \frac{d}{dr} f(r, t) - \frac{1}{r^3} f(r, t) \right] \\
 &= \vec{a} \left[\frac{1}{r^2} \frac{d^2}{dr^2} f(r, t) - \frac{3}{r^3} \frac{d}{dr} f(r, t) + \frac{3}{r^4} f(r, t) \right. \\
 &\quad \left. + 3 \frac{1}{r^2} \frac{d}{dr} f(r, t) - \frac{1}{r^3} f(r, t) \right] = \vec{a} \frac{1}{r} \frac{d^2}{dr^2} f(r, t)
 \end{aligned} \tag{A-7}$$

Therefore $\nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) + k^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = 0$ becomes

$$\bar{\mathbf{a}} \frac{1}{r} \frac{d^2}{dr^2} f(r, t) + k^2 \bar{\mathbf{a}} \frac{f(r, t)}{r} = 0 \quad [\text{A-8a}]$$

or
$$\frac{d^2}{dr^2} f(r, t) + k^2 f(r, t) = 0 \quad [\text{A-8b}]$$

Therefore, we see that the homogeneous Helmholtz equation has two independent solutions -- viz.

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}, t) = \bar{\mathbf{a}} C_+ \frac{\exp[-jkr]}{r} + \bar{\mathbf{a}} C_- \frac{\exp[+jkr]}{r} \quad [\text{A-9}]$$

$$\bar{\mathbf{a}} C_+ \frac{\exp[-jkr]}{r} \quad \text{Outwardly propagating spherical wave}$$

$$\bar{\mathbf{a}} C_- \frac{\exp[+jkr]}{r} \quad \text{Inwardly propagating spherical wave}$$

2. To determine the constant in the outwardly propagating spherical wave, **we now study the behavior of the inhomogeneous equation in the vicinity of the singularity** -- i.e. at $r = 0$ -- where the source of the wave must be located. To cope with the singularity, we integrate the inhomogeneous equation over a small sphere of radius R centered at $r = 0$.

$$\int_{\text{vol. of sphere}} \nabla^2 \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) dV + k^2 \int_{\text{vol. of sphere}} \vec{\mathbf{A}}(\vec{\mathbf{r}}, t) dV = -\mu \int_{\text{vol. of sphere}} \vec{\mathbf{J}}(\vec{\mathbf{r}}, t) dV \quad [\text{A-10}]$$

If we use Gauss' theorem to transform the first term on the left hand side

$$\int_{\text{surf. of sphere}} [d\vec{S} \cdot \vec{A}(\vec{r}, t) + k^2 \vec{A}(\vec{r}, t)] dV = -\mu \int_{\text{vol. of sphere}} \vec{J}(\vec{r}, t) dV \quad [A-11]$$

and substitute the outwardly propagating spherical wave form

$$\int_{\text{surf. of sphere}} \left[d\vec{S} \cdot \vec{r} \right] \frac{d}{dr} C_+ \frac{\exp[-jkr]}{r} + k^2 \int_{\text{vol. of sphere}} C_+ \frac{\exp[-jkr]}{r} dV = -\mu \int_{\text{vol. of sphere}} [\vec{a} \cdot \vec{J}(\vec{r}, t)] dV \quad [A-12a]$$

$$C_+ \int_{\text{surf. of sphere}} \frac{d}{dr} \frac{\exp[-jkr]}{r} R^2 d\Omega + k^2 \int_{\text{vol. of sphere}} C_+ \frac{\exp[-jkr]}{r} dV = -\mu \int_{\text{vol. of sphere}} [\vec{a} \cdot \vec{J}(\vec{r}, t)] dV \quad [A-12b]$$

Therefore in the limit that the radius of the small sphere goes to zero

$$-4 C_+ = -\mu \int_{\text{vol. of sphere}} [\vec{a} \cdot \vec{J}(\vec{r}, t)] dV \quad [A-13]$$

R → 0

so that

$$\vec{A}(\vec{r}, t) = \frac{\mu}{4} \int_{\text{vol. of sphere}} [\vec{J}(\vec{r}, t)] dV \frac{\exp[-jkr]}{r}$$

$$[A-14]$$