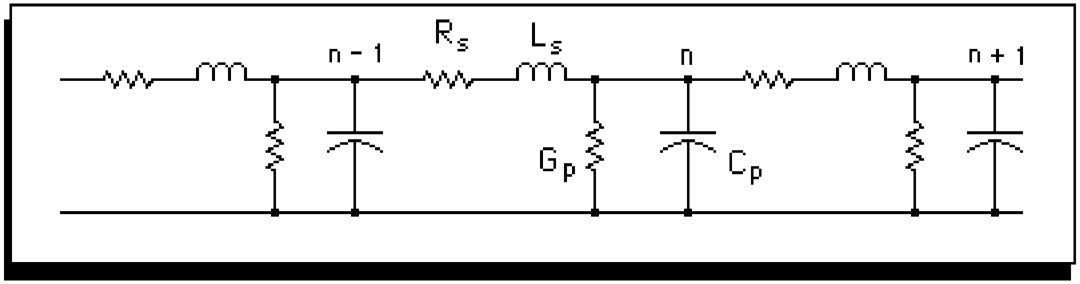
TRANSMISSION LINE THEORY

I. The Transmission Line Model:

Consider the following repeating (uniform) sequence of "lumped" circuit elements:



Applying elementary circuit analysis to each node of such a "discrete" transmission line we may write a set of basic circuit equations.

$$v_{n+1}(t) = v_n(t) - \mathrm{R_s}\, i_{n+1}(t) - \mathrm{L_s} \frac{d}{dt} i_{n+1}(t) \qquad \text{[I-1a]}$$

$$i_{n+1}(t) = i_n(t) - \mathrm{G_p}\, v_n(t) - \mathrm{C_p} \frac{d}{dt} v_n(t) \qquad \text{[I-1b]}$$

The crucial matter is that the voltage and current vary both in time and space! To obtain a solution, we first deal with the time dependence by making use of the "phasor" concept -- *i.e.* we replace the time dependent variables with their Fourier Transforms

$$v_n(t) = \int_{-\infty}^{+\infty} V_n(\omega) \exp[j\,\omega t]\, d\omega \quad \text{and} \quad i_n(t) = \int_{-\infty}^{+\infty} I_n(\omega) \exp[j\,\omega t]\, d\omega \qquad \text{[I-2]}$$

or in the language of circuit analysis

$$v_n(t) = \quad \{V_n(\omega) \exp[j\,\omega t]\} = |V_n(\omega)| \cos(\omega t + \phi_V) \qquad \text{[I-3a]}$$

$$i_n(t) = \quad \{I_n(\omega) \exp[j\,\omega t]\} = |I_n(\omega)| \cos(\omega t + \phi_I) \qquad \text{[I-3b]}$$

Thus, the set of differential circuit equations for a discrete, uniform transmission line becomes a huge set of algebraic equations -- *viz*.

$$V_{n+1}(\omega) = V_n(\omega) - Z_s(\omega)\, I_{n+1}(\omega) \qquad [\text{ I-4a }]$$

$$I_{n+1}(\omega) = I_n(\omega) - Y_p(\omega)\, V_n(\omega) \qquad [\text{ I-4b }]$$

where $Z_s(\omega) = R_s + j\,\omega L_s$ and $Y_p(\omega) = G_p + j\,\omega C_p$ are, respectively, the ***series impedance*** and the ***shunt (parallel) admittance*** of the transmission line.

II. Exact Solutions of Transmission Line Equations:

Our task is to solve Eqs. [I-4]. To that end, we first cast this array of coupled **inhomogeneous** equations in the form of a set of coupled, **homogeneous** algebraic equations -- *viz*.

$$Z_s(\omega)\, Y_p(\omega)\, V_n(\omega) = V_{n+1}(\omega) + V_{n-1}(\omega) - 2\, V_n(\omega) \qquad [\text{ II-1a }]$$

$$Z_s(\omega)\, Y_p(\omega)\, I_n(\omega) = I_{n+1}(\omega) + I_{n-1}(\omega) - 2\, I_n(\omega) \qquad [\text{ II-1b }]$$

Fortunately, here is an amazingly simple set of solutions for this enormous set of algebraic equations. These solutions may be written in the form

$$V_n(\omega) = \{\text{a complex constant}\} \exp\left[j\, n\, \varphi(\omega)\right] \qquad [\text{ II-2a }]$$

$$I_n(\omega) = \{\text{another complex constant}\} \exp\left[j\, n\, \varphi(\omega)\right] \qquad [\text{ II-2b }]$$

We might characterize these solutions as *constant phase solutions* in the sense that the solution at a given node along transmission line is identical to the solution at an adjacent node except for constant phase factor. If these constant phase solutions are to be valid solutions of Eqs. [II-1], the phase constant $\varphi(\omega)$ must satisfy the equation

$$Z_s(\omega)\, Y_p(\omega) \exp\left[j\, n \varphi(\omega)\right] = \exp\left[j\, (n+1)\, \varphi(\omega)\right] + \exp\left[j\, (n-1)\, \varphi(\omega)\right] - 2 \exp\left[j\, n\, \varphi(\omega)\right] \qquad [\text{ II-3 }]$$

Canceling the common $\exp[j\,n\varphi(\omega)]$ factor on both sides of the equation, we obtain

$$\begin{aligned} Z_s(\omega)\,Y_p(\omega) &= \exp[j\,\varphi(\omega)] + \exp[-j\,\varphi(\omega)] - 2 \\ &= \{\exp[j\,\varphi(\omega)/2] - \exp[-j\,\varphi(\omega)/2]\}^2 = \{2j\sin[\varphi(\omega)/2]\}^2 \end{aligned} \qquad \text{[II-4]}$$

Thus, we have obtained an extremely important result which we will, hereafter, refer to as the **dispersion relationship for a discrete, uniform transmission line** -- *viz.*

$$\boxed{2j\sin[\varphi(\omega)/2] = \pm\sqrt{Z_s(\omega)\,Y_p(\omega)}} \qquad \text{[II-5]}$$

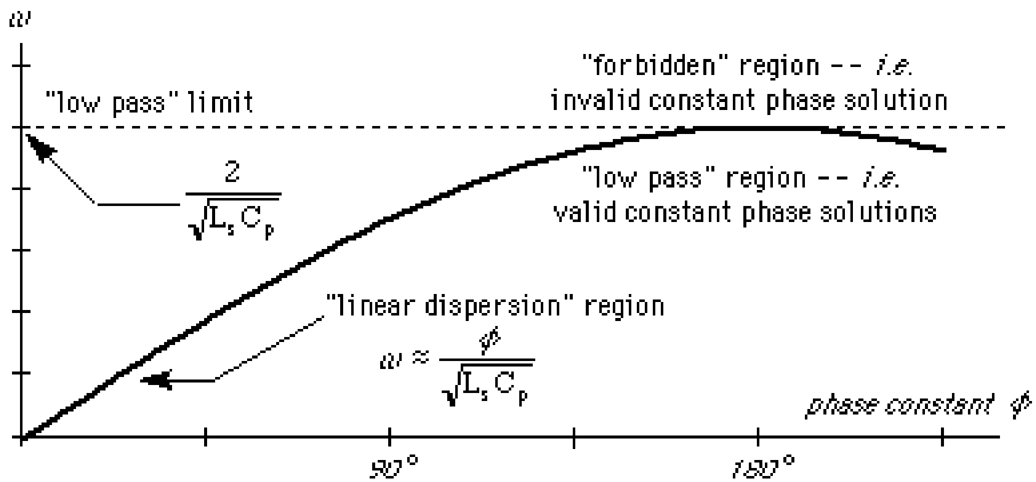
Important Special Cases:

1. The "Ideal" or "Lossless" LC-Transmission Line:

   If we take $Z_s(\omega) = j\,\omega L_s$ and $Y_p(\omega) = j\,\omega C_p$, then Eq [II-5] becomes

$$\sin(\varphi(\omega)/2) = \omega\sqrt{L_s\,C_p}/2 \qquad \text{[II-6]}$$

   which is the **dispersion relationship of a discrete, uniform, ideal transmission line** (Note that the discrete ideal line is, effectively, a low-pass filter.).



2. The "Lossy" RC-Transmission Line:

If we take $Z_s(\omega) = R_s$ and $Y_p(\omega) = j\,\omega C_p$, then Eq [II-5] becomes

$$\sin(\varphi(\omega)/2) = [2j]^{-1}\sqrt{j\,\omega R_s C_p} \qquad \text{[II-7]}$$

We have a problem! What, in heavens name, do we mean by the square root of "j" (*i.e.* the fourth root of -1)? To interpret what is meant by $\sqrt{j}$, note that $j = \left\{\frac{1+j}{\sqrt{2}}\right\}^2$ so that

$$\boxed{\sqrt{j} = \pm\left\{\frac{1+j}{\sqrt{2}}\right\}} \qquad \text{[II-8]}$$

Therefore, the dispersive relationship for a "lossy" RC-transmission line -- *i.e.* Eq. [II-7] becomes

$$\sin\left(\frac{\varphi}{2}\right) = \sin\left(\frac{\varphi' + j\varphi''}{2}\right) = \sin\left(\frac{\varphi'}{2}\right)\cos\left(\frac{j\varphi''}{2}\right) + \cos\left(\frac{\varphi'}{2}\right)\sin\left(\frac{j\varphi''}{2}\right)$$
$$= \pm\frac{\{1-j\}}{2}\sqrt{\frac{\omega R_s C_p}{2}} \qquad \text{[II-9]}$$

which yields, upon equating real and imaginary parts,

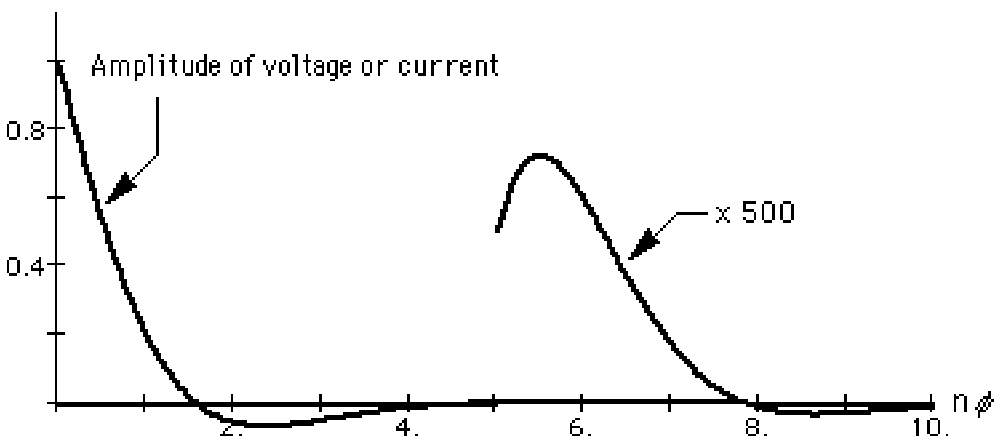
$$\sin\left(\frac{\varphi'}{2}\right)\cosh\left(\frac{\varphi''}{2}\right) = \pm\frac{1}{2}\sqrt{\frac{\omega R_s C_p}{2}} \qquad \text{[II-10a]}$$

$$\cos\left(\frac{\varphi'}{2}\right)\sinh\left(\frac{\varphi''}{2}\right) = \mp\frac{1}{2}\sqrt{\frac{\omega R_s C_p}{2}}\ . \qquad \text{[II-10b]}$$

For ease of interpretation, we make the small argument approximation so that

$$\varphi' \approx \pm\sqrt{\frac{\omega R_s C_p}{2}} \qquad \leftarrow \text{Phase shift per section} \qquad \text{[II-11a]}$$

$$\varphi'' = \mp\sqrt{\frac{\omega R_s C_p}{2}} \qquad \leftarrow \text{Attenuation per section} \qquad \text{[II-11b]}$$



III. Continuous Transmission Lines

An Approximate Solution of Transmission Line Equations:

In most instances, we are interested in continuous rather than discrete transmission lines. To obtain a representation of the voltage across and current along a continuous line, we develop a "continuous approximation" of the basic circuit equations by making use of a Taylor expansion for small node spatial separation. Before looking at the most general case, it is useful to first look at the lossless or ideal case. From Eqs. [I-1] we may write the basic circuit equations

$$v_{n+1}(t) = v_n(t) - \mathrm{L_s} \frac{d}{dt}\, i_{n+1}(t) \qquad \text{[III-1a]}$$

$$i_{n+1}(t) = i_n(t) - \mathrm{C_p} \frac{d}{dt}\, v_n(t) \qquad \text{[III-1b]}$$

Using the Taylor expansion for small node spatial separation, Δz, we obtain

$$v_{n+1}(t) \approx v_n(t) + \left(z_{n+1} - z_n\right) \frac{\partial}{\partial z} v_n(t) = v_n(t) + \Delta z \frac{\partial}{\partial z} v_n(t) \qquad \text{[III-2a]}$$

$$i_{n+1}(t) \approx i_n(t) + \left(z_{n+1} - z_n\right) \frac{\partial}{\partial z} i_n(t) = i_n(t) + \Delta z \frac{\partial}{\partial z} i_n(t) \qquad \text{[III-2b]}$$

Therefore Eqs [III-1a] and [III-1b] become

$$v_n(t) + \Delta z \frac{\partial}{\partial z} v_n(t) \approx v_n(t) - \mathrm{L_s} \frac{d}{dt} \left\{ i_n(t) + \Delta z \frac{\partial}{\partial z} i_n(t) \right\} \qquad \text{[III-3a]}$$

$$i_n(t) + \Delta z \frac{\partial}{\partial z} i_n(t) \approx i_n(t) - \mathrm{C_p} \frac{d}{dt}\, v_n(t) \qquad \text{[III-3b]}$$

To first-order in Δz, we obtain the famous inhomogeneous **"telegrapher equations"** for a lossless transmission line -- *viz*.

$$\frac{\partial}{\partial z} v(z,t) \approx -\mathrm{l_s} \frac{\partial}{\partial t} i(z,t) \qquad \text{where} \qquad \mathrm{l_s} = \lim_{\Delta z \to 0} \left(\frac{L_s}{\Delta z} \right) \qquad \text{[III-4a]}$$

$$\frac{\partial}{\partial z} i(z,t) \approx -\mathrm{c_p} \frac{\partial}{\partial t} v(z,t) \qquad \text{where} \qquad \mathrm{c_p} = \lim_{\Delta z \to 0} \left(\frac{C_p}{\Delta z} \right) \qquad \text{[III-4b]}$$

which, in turn, yields the even more famous homogeneous **"wave equations"**

$$\frac{\partial^2}{\partial z^2} v(z,t) = -\mathrm{l_s} \frac{\partial}{\partial z} \frac{\partial}{\partial t} i(z,t) = -\mathrm{l_s} \frac{\partial}{\partial t} \frac{\partial}{\partial z} i(z,t) = \mathrm{l_s c_p} \frac{\partial^2}{\partial t^2} v(z,t)$$

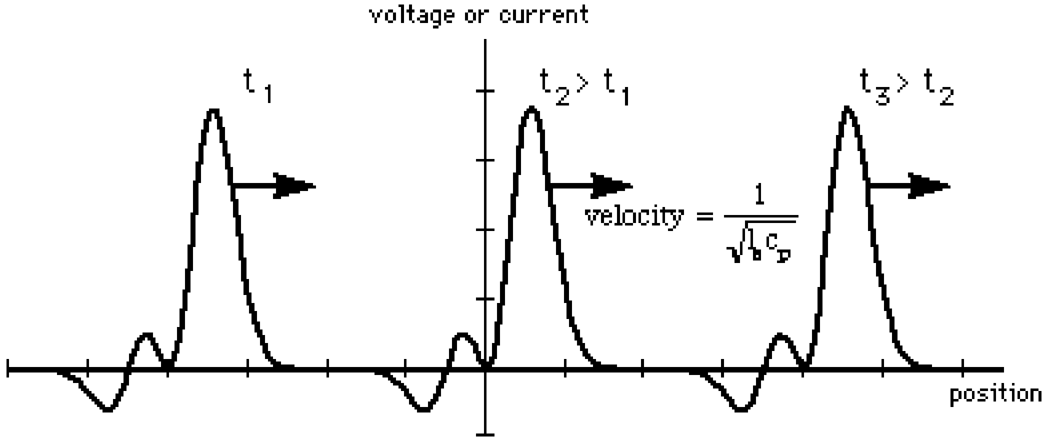
or

$$\frac{\partial^2}{\partial z^2} v(z,t) = \mathrm{l_s c_p} \frac{\partial^2}{\partial t^2} v(z,t) \qquad \text{[III-5a]}$$

and

$$\frac{\partial^2}{\partial z^2} i(z,t) = \mathrm{l_s c_p} \frac{\partial^2}{\partial t^2} i(z,t) \qquad \text{[III-5b]}$$

The truly remarkable point is <u>that any old function</u> of the form $v(z,t) = \mathrm{f}(z - \mathrm{v}\,t)$ and/or of the form $v(z,t) = \mathrm{g}(z + \mathrm{v}\,t)$ will satisfy the Telegrapher and Wave equations!!!!



Wave equation(s):

$$\frac{\partial^2}{\partial z^2} v(z,t) = \mathrm{l_s c_p} \frac{\partial^2}{\partial t^2} v(z,t) \quad \rightarrow \quad \mathrm{f}''(z - \mathrm{v}\,t) = \mathrm{l_s c_p}\,(-\mathrm{v})^2\,\mathrm{f}''(z - \mathrm{v}\,t) \qquad \text{[III-6a]}$$

$$\frac{\partial^2}{\partial z^2} v(z,t) = \mathrm{l_s c_p} \frac{\partial^2}{\partial t^2} v(z,t) \quad \rightarrow \quad \mathrm{g}''(z + \mathrm{v}\,t) = \mathrm{l_s c_p}\,(+\mathrm{v})^2\,\mathrm{g}''(z + \mathrm{v}\,t) \qquad \text{[III-6b]}$$

Telegrapher equations:

$$\frac{\partial}{\partial z} i(z,t) = -\mathrm{c_p} \frac{\partial}{\partial t} v(z,t) = -\mathrm{c_p}\,(-\mathrm{v})\,\mathrm{f}'(z - \mathrm{v}\,t) \quad \rightarrow \quad i_+(z,t) = \mathrm{c_p}\,\mathrm{v}\,\mathrm{f}(z - \mathrm{v}\,t) \qquad \text{[III-7a]}$$

$$\frac{\partial}{\partial z} i(z,t) = -\mathrm{c_p} \frac{\partial}{\partial t} v(z,t) = -\mathrm{c_p}\,(+\mathrm{v})\,\mathrm{g}'(z + \mathrm{v}\,t) \quad \rightarrow \quad i_-(z,t) = -\mathrm{c_p}\,\mathrm{v}\,\mathrm{g}(z + \mathrm{v}\,t) \qquad \text{[III-7b]}$$

so that

$$\mathrm{v} = \frac{1}{\sqrt{\mathrm{l_s c_p}}} \qquad \text{is the wave velocity} \qquad \text{[III-8a]}$$

$$z_c = \frac{1}{c_p\ v} = \sqrt{\frac{l_s}{c_p}} \qquad \text{is the characteristic impedance} \qquad [\text{ III-8b }]$$

General Uniform, Continuous Transmission Line:

We now turn to the more general case represented by Eqs. [I-4]. Again, we develop a "continuous approximation" of these equations by making use of a Taylor expansion for small spatial separation of the nodes. Thus, Eqs. [I-4] become

$$V_n(\omega) + Z_s(\omega)\ I_n(\omega) = V_{n-1}(\omega) \approx V_n(\omega) - \Delta z\ \frac{\partial}{\partial z} V_n(\omega) \qquad [\text{ III-9a }]$$

$$I_n(\omega) - Y_p(\omega)\ V_n(\omega) = I_{n+1}(\omega) \approx I_n(\omega) + \Delta z\ \frac{\partial}{\partial z} I_n(\omega) \qquad [\text{ III-9b }]$$

Once again, to first-order in Δz, these gyrations lead to a more general version of the inhomogeneous Telegrapher equations -- *viz*.

$$\frac{\partial}{\partial z} V(z,\omega) \approx -z_s\ I(z,\omega) \qquad \text{where} \qquad z_s = \lim_{\Delta z \to 0} \left(\frac{Z_s}{\Delta z}\right) \qquad [\text{ III-10a }]$$

$$\frac{\partial}{\partial z} I(z,\omega) \approx -y_p\ V(z,\omega) \qquad \text{where} \qquad y_p = \lim_{\Delta z \to 0} \left(\frac{Y_p}{\Delta z}\right) \qquad [\text{ III-10b }]$$

and to a more general version of the homogeneous Helmhotz equation(s) -- *viz*.

$$\frac{\partial^2}{\partial z^2} V(z,\omega) = z_s y_p\ V(z,\omega) \qquad [\text{ III-11a }]$$

$$\frac{\partial^2}{\partial z^2} I(z,\omega) = z_s y_p\ I(z,\omega) \qquad [\text{ III-11b }]$$

Drawing on our experience above in the analysis of the discrete case, we now look for solutions in the form

$$V(z,\omega) = V_o \exp[\gamma(\omega)\, z] \quad \text{and} \quad I(z,\omega) = I_o \exp[\gamma(\omega)\, z] \qquad \text{[III-12]}$$

so that

$$\gamma(\omega)\, V(z,\omega) = -z_s\, I(z,\omega) \quad \text{and} \quad \gamma(\omega)\, I(z,\omega) = -y_p\, V(z,\omega) \qquad \text{[III-13]}$$

and

$$\gamma^2(\omega)\, V(z,\omega) = z_s y_p\, V(z,\omega) \quad \text{and} \quad \gamma^2(\omega)\, I(z,\omega) = z_s y_p\, I(z,\omega) \qquad \text{[III-14]}$$

where $\gamma(\omega)$ is, in general, complex -- *i.e.* $\gamma(\omega) = \alpha(\omega) + j\beta(\omega)$. Therefore, if the proposed solution is to valid we must have

$$\gamma^2(\omega) = [\alpha(\omega) + j\,\beta(\omega)]^2 = z_s(\omega)\, y_p(\omega) = [r_s + j\,\omega l_s]\,[g_p + j\,\omega c_p] \qquad \text{[III-15]}$$

On equating real and imaginary parts of this expression, we obtain

$$\alpha^2(\omega) - \beta^2(\omega) = r_s\, g_p - \omega^2\, l_s\, c_p \qquad \text{[III-16a]}$$

and

$$2\,\alpha(\omega)\,\beta(\omega) = \omega[r_s c_p + g_p l_s]\ . \qquad \text{[III-16b]}$$

In the small attenuation approximation, we see that the phase shift per unit is given by

$$\beta(\omega) \approx \omega\sqrt{l_s c_p}\sqrt{\left\{1 - \frac{r_s}{\omega\, l_s}\,\frac{g_p}{\omega\, c_p}\right\}} \qquad \text{[III-17a]}$$

and the attenuation per unit length by

$$\alpha(\omega) \approx \frac{1}{2}\left\{r_s\sqrt{\frac{c_p}{l_s}} + g_p\sqrt{\frac{l_s}{c_p}}\right\} \qquad \text{[III-17b]}$$

If we consider, once again, the all important special case of a "lossless" LC transmission line -- *i.e.* where $g_p \equiv r_s \equiv 0$, we see that

$$\gamma(\omega) = \alpha(\omega) + j\,\beta(\omega) = 0 \pm j\,\omega\sqrt{\mathrm{l_s c_p}} \qquad [\text{ III-18 }]$$

Thus, for a **general uniform, continuous transmission line**, the linear combination of **two independent frequency domain** solutions may be written

$$V(z,\omega) = V_+(\omega)\,\exp\left[-\gamma(\omega)\,z\right] + V_-(\omega)\,\exp\left[+\gamma(\omega)\,z\right] \qquad [\text{ III-19 }]$$

where $\gamma(\omega) = \sqrt{\mathrm{z_s}(\omega)\,\mathrm{y_p}(\omega)}$. In the time domain for a single frequency, we have

$$v(z,t) = \left\{V_+(\omega)\,\exp\left[-\gamma(\omega)\,z + j\,\omega t\right] + V_-(\omega)\,\exp\left[+\gamma(\omega)\,z + j\,\omega t\right]\right\} \qquad [\text{ III-20 }]$$

To be concrete and for ease of interpretation, we discuss in much of what follows a general lossless or non-attenuating transmission line so that the time domain solution becomes

$$\begin{aligned} v(z,t) &= \left(V_+(\omega)\,\exp\left\{j\left[\omega t - \beta(\omega) z\right]\right\} + V_-(\omega)\,\exp\left\{j\left[\omega t + \beta(\omega) z\right]\right\}\right) \\ &= \left|V_+(\omega)\right|\cos\left[\omega t - \beta(\omega) z + \varphi_+\right] + \left|V_-(\omega)\right|\cos\left[\omega t + \beta(\omega) z + \varphi_-\right] \end{aligned} \qquad [\text{ III-21 }]$$

As in earlier discussions, we interpret

$$\left|V_+(\omega)\right|\cos\left[\omega t - \beta(\omega) z + \varphi_+\right]$$

as a continuous wave propagating to the right (positive z-direction) and

$$\left|V_-(\omega)\right|\cos\left[\omega t + \beta(\omega) z + \varphi_-\right]$$

as a continuous wave propagating to the left (negative z-direction). From one of the Telegrapher equations -- *viz.* Eq. [III-10a] -- we see that

$$I\,(z,\omega) = -\frac{1}{\mathrm{z_s}(\omega)}\,\frac{\partial}{\partial z} V\,(z,\omega) \qquad [\text{ III-22a }]$$

so that

$$I(z,\omega) = I_+(\omega)\exp[-\gamma(\omega)z] + I_-(\omega)\exp[+\gamma(\omega)z]$$
$$= \frac{1}{Z_c(\omega)}\{V_+(\omega)\exp[-\gamma(\omega)z] - V_-(\omega)\exp[+\gamma(\omega)z]\} \qquad \text{[III-22b]}$$

where $$Z_c(\omega) = z_s(\omega)/\gamma(\omega) = \sqrt{z_s(\omega)/y_p(\omega)} \qquad \text{[III-23]}$$

is the **characteristic impedance** of the transmission line. For a "lossless" LC transmission line, we see that

$$Z_c(\omega) = Z_c = \sqrt{l_s/c_p} \qquad \text{[III-24]}$$

Finally, we introduce the extreme important (but rather confusing) notion of a spatially varying **wave impedance** which is define as

$$Z(z,\omega) \equiv \frac{V(z,\omega)}{I(z,\omega)}$$
$$= Z_c(\omega)\left\{\frac{V_+(\omega)\exp[-\gamma(\omega)z] + V_-(\omega)\exp[+\gamma(\omega)z]}{V_+(\omega)\exp[-\gamma(\omega)z] - V_-(\omega)\exp[+\gamma(\omega)z]}\right\} \qquad \text{[III-25]}$$
$$= Z_c(\omega)\left\{\frac{1 + [V_-(\omega)/V_+(\omega)]\exp[+2\gamma(\omega)z]}{1 - [V_-(\omega)/V_+(\omega)]\exp[+2\gamma(\omega)z]}\right\} = Z_c(\omega)\left\{\frac{1 + \Gamma_V(z,\omega)}{1 - \Gamma_V(z,\omega)}\right\}$$

where $$\Gamma_V(z,\omega) \equiv [V_-(\omega)/V_+(\omega)]\exp[+2\gamma(\omega)z] \qquad \text{[III-26]}$$

is the spatial varying **voltage reflection coefficient**. Thus, we may write the general solution in the compact form

$$V(z,\omega) = V_+(\omega)\exp[-\gamma(\omega)z]\{1 + \Gamma_V(z,\omega)\} \qquad \text{[III-27a]}$$

$$I(z,\omega)\,Z_c(\omega) = V_+(\omega)\exp[-\gamma(\omega)z]\{1 - \Gamma_V(z,\omega)\} \qquad \text{[III-27b]}$$

IV. Transmission Lines Terminations:

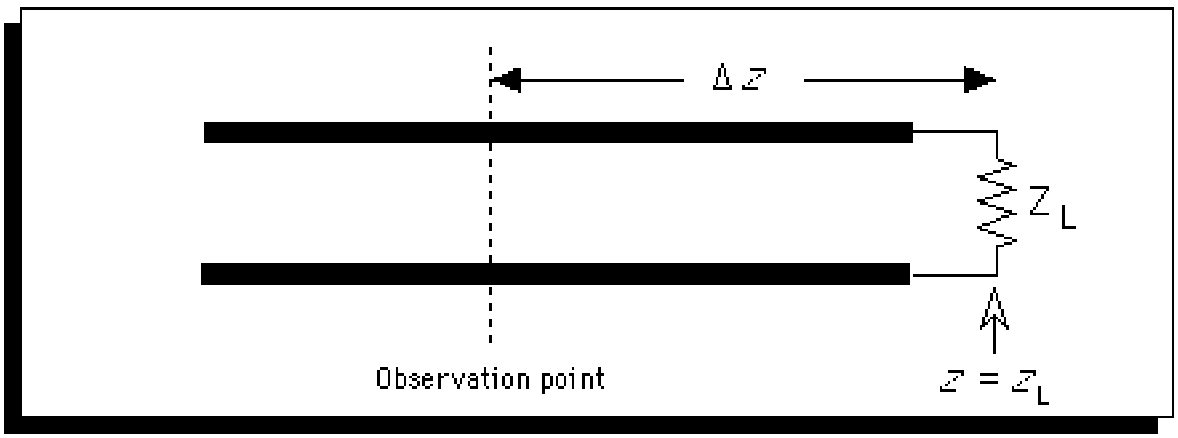
It remains for us to determine the spatial varying wave impedance and reflection coefficient which must satisfy the dual equations

$$Z(z,\omega) = Z_c(\omega)\left\{\frac{1+\Gamma_V(z,\omega)}{1-\Gamma_V(z,\omega)}\right\} \quad \text{[IV-1a]}$$

$$\Gamma_V(z,\omega) = \frac{Z(z,\omega) - Z_c(\omega)}{Z(z,\omega) + Z_c(\omega)} \quad \text{[IV-1b]}$$

at every point along the transmission line. To that end we must consider the effect of transmission line terminations.

LOADED TRANSMISSION LINE



Consider first some simple, but important cases.

1. A "shorted" transmission line -- *i. e.* $Z(z_L,\omega) = 0$ so that $\Gamma_V(z_L,\omega) = -1$.

At other points along the line

$$\Gamma_V(z,\omega) = \left[V_-(\omega)/V_+(\omega)\right] \exp\left[+2\gamma(\omega)\, z\right] = \Gamma_V\left(z_L,\omega\right) \exp\left\{+2\gamma(\omega)\left[z - z_L\right]\right\} \qquad \text{[IV-2]}$$

so that in this case

$$\Gamma_V(z,\omega) = -\exp\left\{-2\gamma(\omega)\left[z_L - z\right]\right\} \qquad \text{[IV-3a]}$$

and

$$\begin{aligned} V(z,\omega) &= V_+(\omega) \exp\left[-\gamma(\omega) z\right]\left(1 - \exp\left\{-2\gamma(\omega)\left[z_L - z\right]\right\}\right) \\ &= V_+(\omega) \exp\left[-\gamma(\omega) z - \gamma(\omega)\left[z_L - z\right]\right]\left(\exp\left\{+\gamma(\omega)\left[z_L - z\right]\right\} - \exp\left\{-\gamma(\omega)\left[z_L - z\right]\right\}\right) \qquad \text{[IV-3b]} \\ &= V_+(\omega) \exp\left[-\gamma(\omega) z_L\right]\left(\exp\left\{+\gamma(\omega)\left[z_L - z\right]\right\} - \exp\left\{-\gamma(\omega)\left[z_L - z\right]\right\}\right) \end{aligned}$$

When the attenuation is zero

$$V(z,\omega) = 2\, j\, V_+(\omega) \exp\left[-j\beta(\omega)\, z_L\right] \sin\left\{\beta(\omega)\left[z_L - z\right]\right\} \qquad \text{[IV-4a]}$$

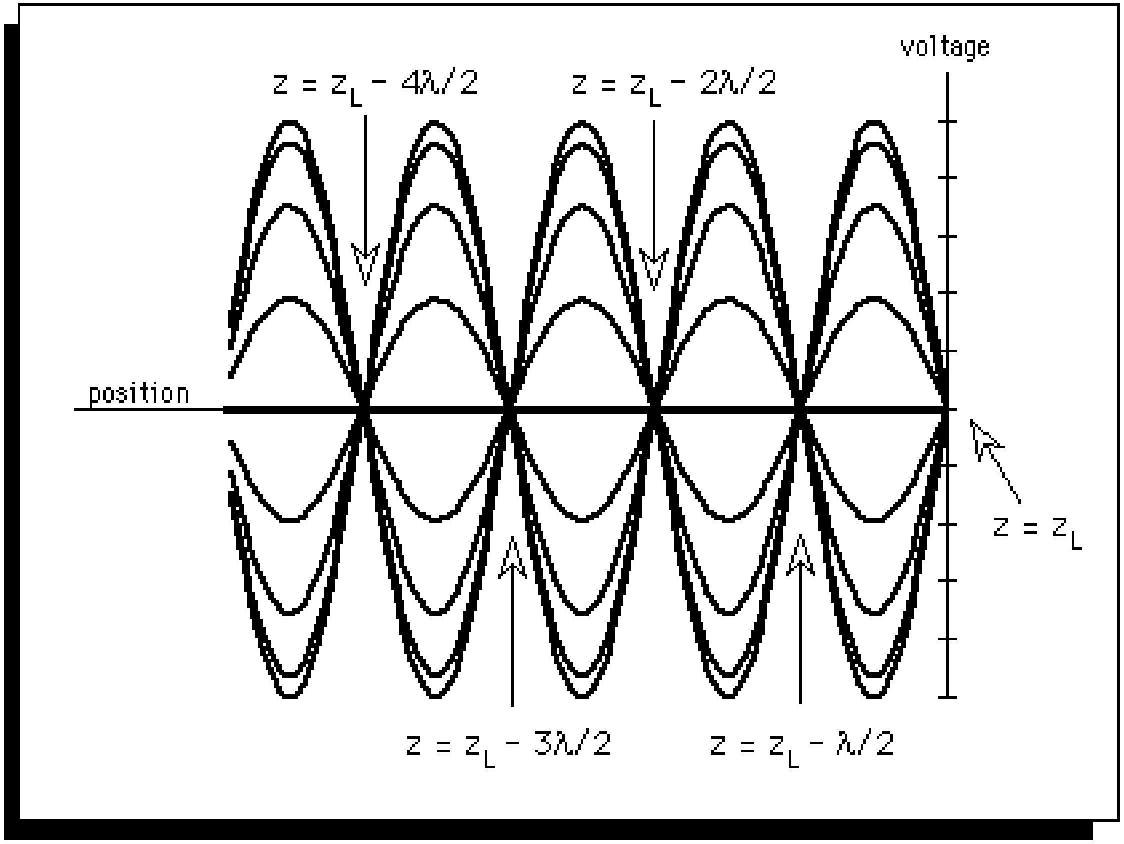
$$Z_c(\omega)\, I(z,\omega) = 2\, V_+(\omega) \exp\left[-j\,\beta(\omega)\, z_L\right] \cos\left\{\beta(\omega)\left[z_L - z\right]\right\} \qquad \text{[IV-4b]}$$

$$Z(z,\omega) = \frac{V(z,\omega)}{I(z,\omega)} = j\, Z_c(\omega) \tan\left\{\beta(\omega)\left[z_L - z\right]\right\} \qquad \text{[IV-4c]}$$

Such a solution is called a "pure" **standing wave**. It is a spatial varying voltage oscillation which may be observed with an oscilloscope. The pattern that would be observed is graphed below as it would be seen at **16 distinct times** equally spaced at 1/16 of a period. The voltage across the short is, of coarse, zero at all times! There is, of course, another voltage node whenever

$$\beta(\omega)\left[z_L - z\right] = 2\pi\left[z_L - z\right]/\lambda(\omega) = \pi\left[\text{integer}\right].$$

VOLTAGE ACROSS A *SHORTED* TRANSMISSION LINE



2. <u>An "open" transmission line</u> -- *i. e.* $Z(z_L,\omega) = \infty$ so that $\Gamma_V(z_L,\omega) = +1$.

$$\Gamma_V(z,\omega) = +\exp\{-2\gamma(\omega)[z_L - z]\} \qquad \text{[IV-5a]}$$

and

$$V(z,\omega) = V_+(\omega)\exp[-\gamma(\omega)z]\left(1 + \exp\{-2\gamma(\omega)[z_L - z]\}\right)$$
$$= V_+(\omega)\exp[-\gamma(\omega)z - \gamma(\omega)[z_L - z]]\left(\exp\{+\gamma(\omega)[z_L - z]\} + \exp\{-\gamma(\omega)[z_L - z]\}\right) \qquad \text{[IV-5b]}$$
$$= V_+(\omega)\exp[-\gamma(\omega)z_L]\left(\exp\{+\gamma(\omega)[z_L - z]\} + \exp\{-\gamma(\omega)[z_L - z]\}\right)$$

When the attenuation is zero

$$V(z,\omega) = 2\,V_+(\omega)\exp[-j\,\beta(\omega)\,z_L]\cos\{\beta(\omega)[z_L - z]\} \qquad \text{[IV-6a]}$$

$$Z_c(\omega)\,I(z,\omega) = -j\,2\,V_+(\omega)\exp[-j\,\beta(\omega)\,z_L]\sin\{\beta(\omega)[z_L - z]\} \qquad \text{[IV-6b]}$$

$$Z(z,\omega) = \frac{V(z,\omega)}{I(z,\omega)} = j\,Z_c(\omega)\cot\{\beta(\omega)[z_L - z]\} \qquad \text{[IV-6c]}$$

Again as above, the solution is a "pure" **standing wave**. However, the "open" line is the dual of the "shorted" line in the sense that the role of current and voltage are reversed.

3. A "matched" transmission line -- *i. e.* $Z(z_L,\omega) = Z_c(\omega)$ so that $\Gamma_V(z_L,\omega) = 0$.

$$\Gamma_V(z,\omega) = 0 \qquad \text{[IV-7a]}$$

and

$$V(z,\omega) = V_+(\omega)\exp[-\gamma(\omega)z] \qquad \text{[IV-7b]}$$

When the attenuation is zero

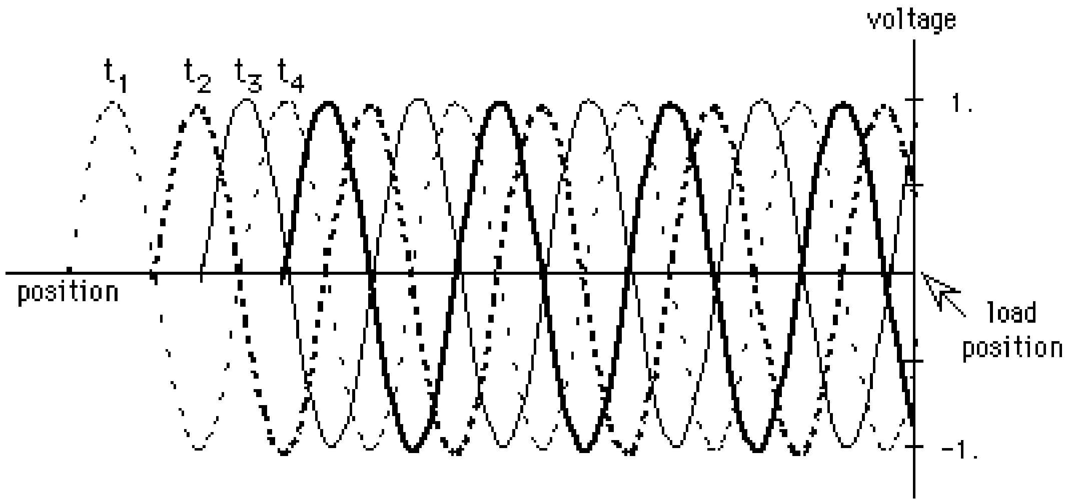
$$V(z,\omega) = V_+(\omega)\exp[-j\beta(\omega)\,z] \qquad \text{[IV-8a]}$$

$$I(z,\omega) = Z_c(\omega)V_+(\omega)\exp[-j\beta(\omega)z] \qquad \text{[IV-8b]}$$

$$Z(z,\omega) = \frac{V(z,\omega)}{I(z,\omega)} = Z_c(\omega) \qquad \text{[IV-8c]}$$

Such a solution is called a "pure" **running wave**. In such cases, the average value (or the amplitude) of the current and voltage are spatially uniform.

VOLTAGE ACROSS A *MATCHED* TRANSMISSION LINE



4. The general case -- *i. e.* $Z\left(z_L,\omega\right) \neq$ any special value.

In these cases the solutions are combinations of **standing** and **running** waves! For nonattenuating lines the, so called, **voltage-standing-wave-ratio** or **VSWR** is an important measure of the character of the solution. From Eq. [III-27a] we see that

$$\left|V\left(z,\omega\right)\right| = \left|V_+(\omega)\right|\left|1 + \Gamma_V\left(z,\omega\right)\right| \qquad \text{[IV-9a]}$$

where

$$\Gamma_V(z,\omega) \equiv \left[V_-(\omega)/V_+(\omega)\right] \exp\left[+2\,j\,\beta(\omega)\,z\right] = \left|\Gamma_V(\omega)\right| \exp\left[j\,\varphi_{\Gamma_V}(z,\omega)\,z\right] \qquad \text{[IV-9b]}$$

Therefore, we have the definition

$$\mathbf{VSWR} \equiv \frac{\left|V(z,\omega)\right|_{\text{Max}}}{\left|V(z,\omega)\right|_{\text{Min}}} = \frac{1+\left|\Gamma_V(\omega)\right|}{1-\left|\Gamma_V(\omega)\right|} \qquad \text{[IV-10]}$$

Note that for a pure running wave solution $\mathbf{VSWR}=1$ and for a pure standing wave solution $\mathbf{VSWR}=\infty$.

We have one last really important task -- *viz.* establishing the all important **wave impedance transformation**. To that end we see from Eqs. [IV-1] that

$$\begin{aligned} Z(z,\omega) &= Z_c(\omega)\left\{\frac{1+\Gamma_V(z,\omega)}{1-\Gamma_V(z,\omega)}\right\} \\ &= Z_c(\omega)\left\{\frac{1+\Gamma_V(z_L,\omega)\exp\{-2\gamma(\omega)[z_L-z]\}}{1-\Gamma_V(z_L,\omega)\exp\{-2\gamma(\omega)[z_L-z]\}}\right\} \qquad \text{[IV-11]} \\ &= Z_c(\omega)\left\{\frac{\exp\{+\gamma(\omega)[z_L-z]\}+\Gamma_V(z_L,\omega)\exp\{-\gamma(\omega)[z_L-z]\}}{\exp\{+\gamma(\omega)[z_L-z]\}-\Gamma_V(z_L,\omega)\exp\{-\gamma(\omega)[z_L-z]\}}\right\} \end{aligned}$$

But $$\Gamma_V(z_L,\omega) = \frac{Z(z_L,\omega)-Z_c(\omega)}{Z(z_L,\omega)+Z_c(\omega)}$$

so that

$$\begin{aligned} Z(z,\omega) &= Z_c(\omega)\left\{\frac{\exp\{+\gamma(\omega)[z_L-z]\}+\dfrac{Z(z_L,\omega)-Z_c(\omega)}{Z(z_L,\omega)+Z_c(\omega)}\exp\{-\gamma(\omega)[z_L-z]\}}{\exp\{+\gamma(\omega)[z_L-z]\}-\dfrac{Z(z_L,\omega)-Z_c(\omega)}{Z(z_L,\omega)+Z_c(\omega)}\exp\{-\gamma(\omega)[z_L-z]\}}\right\} \\ &= Z_c(\omega)\left\{\frac{Z(z_L,\omega)\left[\exp\{+\gamma(\omega)[z_L-z]\}+\exp\{-\gamma(\omega)[z_L-z]\}\right]+Z_c(\omega)\left[\exp\{+\gamma(\omega)[z_L-z]\}-\exp\{-\gamma(\omega)[z_L-z]\}\right]}{Z(z_L,\omega)\left[\exp\{+\gamma(\omega)[z_L-z]\}-\exp\{-\gamma(\omega)[z_L-z]\}\right]+Z_c(\omega)\left[\exp\{+\gamma(\omega)[z_L-z]\}+\exp\{-\gamma(\omega)[z_L-z]\}\right]}\right\} \end{aligned}$$

[IV-12a]

or more compactly

$$Z(z,\omega) = Z_c(\omega)\left\{\frac{Z(z_L,\omega) + Z_c(\omega)\tanh\{\gamma(\omega)[z_L - z]\}}{Z_c(\omega) + Z(z_L,\omega)\tanh\{\gamma(\omega)[z_L - z]\}}\right\} \quad \text{[IV-12b]}$$

For non attenuating lines, this expression reduces to the ***famous impedance transformation formula*** **--** *viz.*

$$\boxed{Z(z,\omega) = Z_c(\omega)\left\{\frac{Z(z_L,\omega) + j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}}{Z_c(\omega) + j\,Z(z_L,\omega)\tan\{\beta(\omega)[z_L - z]\}}\right\}} \quad \text{[IV-13]}$$

Applications of the **famous impedance transformation formula:**

a. "Shorted" transmission line:

$$Z_{\text{short}}(z,\omega) = Z_c(\omega)\left\{\frac{(0) + j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}}{Z_c(\omega) + j\,(0)\tan\{\beta(\omega)[z_L - z]\}}\right\} \quad \text{[IV-14a]}$$

$$= j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}$$

b. "Open" transmission line:

$$Z_{\text{open}}(z,\omega) = Z_c(\omega)\left\{\frac{(\infty) + j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}}{Z_c(\omega) + j\,(\infty)\tan\{\beta(\omega)[z_L - z]\}}\right\} \quad \text{[IV-14b]}$$

$$= -j\,Z_c(\omega)\cot\{\beta(\omega)[z_L - z]\}$$

c. "Matched" transmission line:

$$Z_{\text{match}}(z,\omega) = Z_c(\omega)\left\{\frac{Z_c(\omega) + j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}}{Z_c(\omega) + j\,Z_c(\omega)\tan\{\beta(\omega)[z_L - z]\}}\right\} \qquad \text{[IV-14c]}$$

$$= Z_c(\omega)$$

d. Quarter wavelength matching transformer:

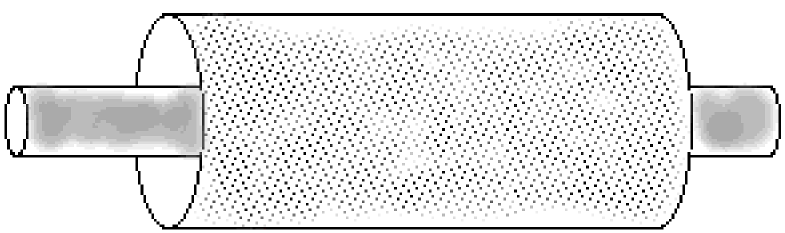
$$Z_{\lambda/4}(z,\omega) = Z_c(\omega)\left\{\frac{Z(z_L,\omega) + j\,Z_c(\omega)\tan\{\pi/2\}}{Z_c(\omega) + j\,Z(z_L,\omega)\tan\{\pi/2\}}\right\} \qquad \text{[IV-14d]}$$

$$= [Z_c(\omega)]^2 / Z(z_L,\omega)$$

Matched if $\quad Z_c(\omega) = \sqrt{Z_{\lambda/4}(z,\omega)\,Z(z_L,\omega)}$!!!

V. Parameters of a Coaxial Transmission Line:

We now look to Maxwell's Equations (in integral form) for values of the line parameters of a coaxial line of inner radius a and outer radius b:



We first make use of the Gaussian law of electrostatics to obtain the capacitance of the line. Assume a Gaussian surface which is an imaginary coaxial cylinder which has a radius r in the range $[a, b]$ and a length $\delta\ell$ so that

$$\iint_S \vec{\mathbf{E}}\;\hat{\mathbf{n}}\;dA = \frac{1}{\varepsilon_0}\iiint_V \rho\;dV \qquad \text{[V-1]}$$

leads to

$$E_r\left[2\pi r\,\delta\ell\right] = -\frac{dV}{dr}\left[2\pi r\,\delta\ell\right] = \frac{Q}{\varepsilon_0} \qquad \text{[V-2]}$$

or

$$dV = -\frac{1}{2\pi\varepsilon_0}\left(\frac{Q}{\delta\ell}\right)\left(\frac{1}{r}\right) \quad \rightarrow \quad V = \frac{1}{2\pi\varepsilon_0}\left(\frac{Q}{\delta\ell}\right)\ln\left(\frac{b}{a}\right) \qquad \text{[V-3]}$$

Therefore, the **capacitance per unit line length** is

$$c_p = \frac{(Q/\delta\ell)}{V} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \qquad \text{[V-4]}$$

Obtaining the inductance of the line is a bit more complicated. We make use of Ampère's law to find the magnetic field and then use Faraday's law to find the induced emf associated with a time varying current. We apply the integral form of Ampère's to a circular loop of radius r which is coaxial with the inner conductor so that

$$\oint_L \vec{\mathbf{B}}\cdot d\hat{\mathbf{l}} = \mu_0 \iint_S \vec{\mathbf{J}}\cdot\hat{\mathbf{n}}\,dA \qquad \text{[V-5]}$$

leads to

$$B_\theta\left[2\pi r\right] = \mu_0 I \quad \text{or} \quad B_\theta = \frac{\mu_0}{2\pi}\left(\frac{I}{r}\right) \qquad \text{[V-6]}$$

We use this expression for the field to find the changing magnetic flux through a loop in the median plane of the coaxial line.

$$\text{emf} = \oint_L \vec{\mathbf{E}} \cdot d\,\hat{\mathbf{l}} = -\frac{\partial}{\partial t}\iint_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\; dA = -\frac{\partial}{\partial t}\left\{ \delta\,\ell \int_a^b B_\theta\, dr \right\}$$
$$= -\frac{\mu_0}{2\pi}\,\delta\,\ell \left\{ \int_a^b \frac{dr}{r} \right\} \frac{\partial I}{\partial t} = -\frac{\mu_0}{2\pi}\,\delta\,\ell\,\left\{ \ln(b/a) \right\} \frac{\partial I}{\partial t} \qquad [\text{ V-7 }]$$

Therefore, the **inductance per unit line length** is

$$l_s = \frac{\mu_0}{2\pi} \ln(b/a) \qquad [\text{ V-8 }]$$

Therefore, Maxwell's equations give us expressions for the all important transmission line parameters of a coaxial line -- *viz*.

$$v = \frac{1}{\sqrt{l_s\, c_p}} = \sqrt{\left[\frac{2\pi}{\mu_0 \ln(b/a)}\right]\left[\frac{\ln(b/a)}{2\pi\,\varepsilon_0}\right]} = \frac{1}{\sqrt{\varepsilon_0\, \mu_0}} = \text{phase velocity} \qquad [\text{ V-9a }]$$

$$Z_c = \sqrt{l_s / c_p} = \sqrt{\left[\frac{\mu_0 \ln(b/a)}{2\pi}\right]\left[\frac{\ln(b/a)}{2\pi\,\varepsilon_0}\right]} = \left[\frac{\ln(b/a)}{2\pi}\right]\sqrt{\frac{\mu_0}{\varepsilon_0}} = \text{characteristic impedance} \qquad [\text{ V-9b }]$$

VI. Jones on Smith Charts:

Let us examine a very important property of the pair of equations [IV-1a] and [IV-1b]. Recall that

$$Z(z,\omega) = Z_c(\omega)\left\{\frac{1 + \Gamma_V(z,\omega)}{1 - \Gamma_V(z,\omega)}\right\} \qquad [\text{ VI-1a }]$$

or

$$z = \frac{1+\Gamma}{1-\Gamma} \qquad \text{where} \qquad z = Z(z,\omega)/Z_c(\omega) \qquad [\text{ VI-1b }]$$

Writing this expression in terms of real and imaginary parts we see that

$$\begin{aligned} z_r + j\,z_i &= \frac{1+\Gamma_r + j\,\Gamma_i}{1-\Gamma_r - j\,\Gamma_i} = \left\{\frac{1+\Gamma_r + j\,\Gamma_i}{1-\Gamma_r - j\,\Gamma_i}\right\}\left\{\frac{1-\Gamma_r + j\,\Gamma_i}{1-\Gamma_r + j\,\Gamma_i}\right\} \\ &= \frac{1-\Gamma_r^{\,2} - \Gamma_i^{\,2} + j\,2\Gamma_i}{(1-\Gamma_r)^2 + (\Gamma_i)^2} \end{aligned} \qquad [\text{ VI-2 }]$$

Equating real and imaginary components on either side of the equation

$$z_r = \frac{1-\Gamma_r^{\,2} - \Gamma_i^{\,2}}{(1-\Gamma_r)^2 + (\Gamma_i)^2} \qquad [\text{ VI-3a }]$$

$$z_i = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + (\Gamma_i)^2} \qquad [\text{ VI-3b }]$$

we obtain

$$1 - 2\Gamma_r + \Gamma_r^{\,2} + \Gamma_i^{\,2} = \frac{1}{z_r}\left\{1 - \Gamma_r^{\,2} - \Gamma_i^{\,2}\right\} \qquad [\text{ VI-4a }]$$

$$1 - 2\Gamma_r + \Gamma_r^{\,2} + \Gamma_i^{\,2} = \frac{1}{z_i}\left\{2\Gamma_i\right\} \qquad [\text{ VI-4b }]$$

or with further *messaging*

$$\Gamma_r^{\,2}\left\{\frac{z_r+1}{z_r}\right\} - 2\Gamma_r + \Gamma_i^{\,2}\left\{\frac{z_r+1}{z_r}\right\} = \frac{1-z_r}{z_r} \qquad [\text{ VI-5a }]$$

$$\Gamma_r^{\,2} - 2\Gamma_r + 1 + \Gamma_i^{\,2} - \frac{2\Gamma_i}{z_i} = 0 \qquad [\text{ VI-5b }]$$

Completing the square in both cases

$$\left\{\Gamma_r^{\,2}-\frac{2z_r}{z_r+1}+\left[\frac{z_r}{z_r+1}\right]^2\right\}+\Gamma_i^{\,2}=\left[\frac{z_r}{z_r+1}\right]^2+\frac{1-z_r}{z_r+1}=\left[\frac{1}{z_r+1}\right]^2 \qquad \text{[VI-6a]}$$

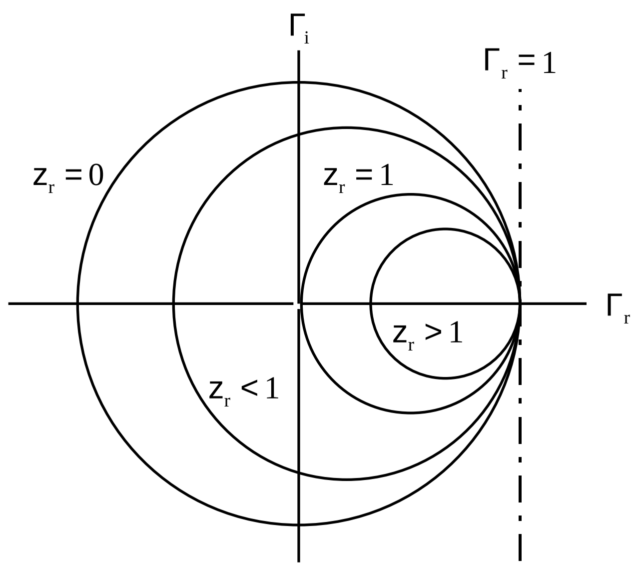
$$\left\{\Gamma_r^{\,2}-2\Gamma_r+1\right\}+\left\{\Gamma_i^{\,2}-\frac{2\Gamma_i}{z_i}+\left[\frac{1}{z_i}\right]^2\right\}=\left[\frac{1}{z_i}\right]^2 \qquad \text{[VI-6b]}$$

Therefore, the loci of constant z_r and constant z_i in the $[\Gamma_r, \Gamma_i]$ plane are equations for circles -- *viz*.

CIRCLES OF CONSTANT "RESISTANCE"

$$\left\{\Gamma_r - \frac{z_r}{z_r+1}\right\}^2 + \Gamma_i^2 = \left\{\frac{1}{z_r+1}\right\}^2 \qquad \text{[VI-7a]}$$

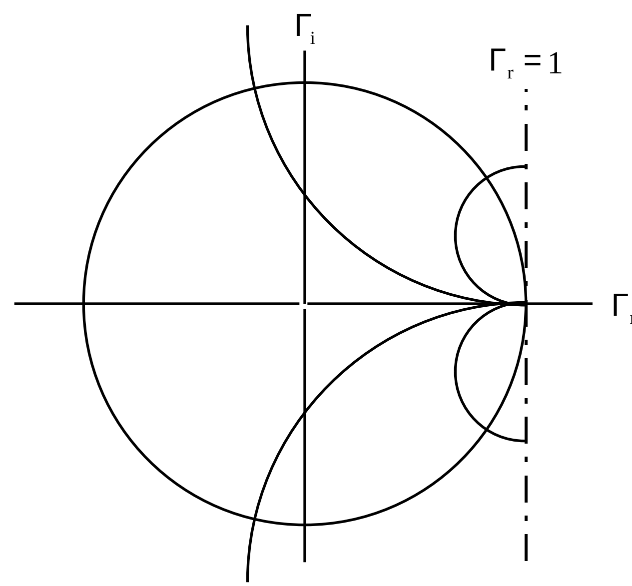
$$\text{radius} = \frac{1}{1+z_r}\,; \quad \text{center} = \left(\frac{z_r}{1+z_r}, 0\right)$$



CIRCLES OF CONSTANT "REACTANCE"

$$\{\Gamma_r - 1\}^2 + \left\{\Gamma_i - \frac{1}{z_i}\right\}^2 = \left\{\frac{1}{z_i}\right\}^2 \qquad \text{[VI-7b]}$$

$$\text{radius} = \frac{1}{z_i}\,; \quad \text{center} = \left(1, \frac{1}{z_i}\right)$$



These isoresistance and isoreactance curves are the basis for the famous and very useful Smith charts.[1]

[1] P. H. Smith, *Electronics* **12**, 29 (1939); **17**, 130 (1944)