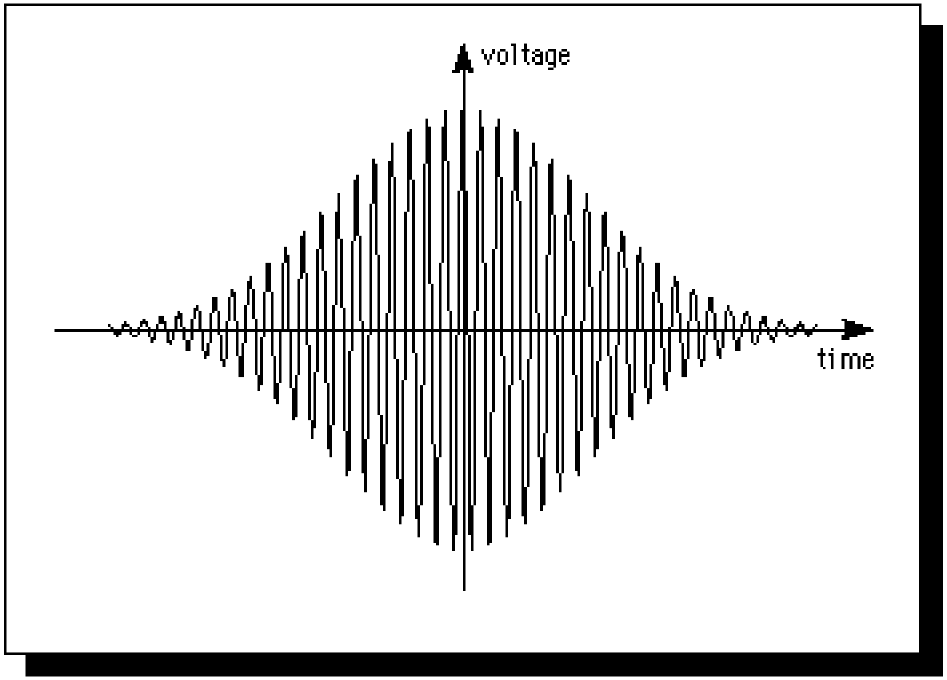
PULSE PROPAGATION ON A
DISPERSIVE TRANSMISSION LINE

I. PRELIMINARIES -- A REVIEW OF SOME BASIC CONCEPTS AND METHODS:

Let us consider the propagation of a typical signaling voltage pulse $V(z, t)$ along a transmission line. At some particular position, the **time variation** of the voltage across the line might look like the following:



Gaussian Pulse

To avoid unnecessary complications, we assume that the **line is matched** -- *i.e.* we consider only traveling waves -- so that we may, in general, write

$$V(z, t) = \int_{-\infty}^{+\infty} V(\omega) \exp\left[-j\ \beta(\omega)\ z\right] \exp\left[j\ \omega\ t\right] d\omega \qquad \text{[I-1]}$$

In most instances we know $\mathrm{V}\left(z_{ref}, t\right)$ at some particular reference position $z = z_{ref}$ along the line, so we may write

$$\mathrm{V}(\omega) = \frac{1}{2\pi} \exp\left[j\,\beta(\omega)\, z_{ref} \right] \int_{-\infty}^{+\infty} \mathrm{V}\left(z_{ref}, t\right) \exp\left[-j\,\omega\, t \right] dt \;\; . \qquad \text{[I-2]}$$

For example, for a Gaussian pulse or "burst" of the form

$$\mathrm{V}_G(0, t) = \mathrm{V}_0 \exp\left[j\,\omega_0\, t \right] \exp\left[-\frac{1}{2} \left(\frac{t}{\tau} \right)^2 \right] . \qquad \text{[I-3a]}$$

at $z = 0$ we know that the frequency spectrum is given by

$$\mathrm{V}_G(\omega) = \mathrm{V}_0 \frac{\tau}{\sqrt{2\pi}} \exp\left\{ -\frac{\left[\tau\,(\omega - \omega_0) \right]^2}{2} \right\} . \qquad \text{[I-3b]}$$

In most cases of interest, the spectrum of the pulse essentially vanishes outside of some relatively narrow frequency range in the vicinity of some *center* frequency, say, ω_0 and, therefore, the pulse may be usefully represented in the time domain as follows:

$$\begin{aligned} \mathrm{V}(z, t) = {} & \exp\left\{ -j\left[\beta(\omega_0)\, z - \omega_0\, t \right] \right\} \\ & \times \int_{-\infty}^{+\infty} \mathrm{V}(\omega - \omega_0) \exp\left\{ -j\left[\beta(\omega) - \beta(\omega_0) \right] z \right\} \exp\left[j\,(\omega - \omega_0)\, t \right] d(\omega - \omega_0) \end{aligned} \qquad \text{[I-4]}$$

As we has seen in earlier discussion, **ideal** or **nondispersive** transmission lines have linear dispersion relationships -- *i.e.* $\beta(\omega) = \omega / v_p$ where v_p is a frequency independent **phase velocity** -- and, thus,

$$\mathrm{V}(z,t) = \exp\left\{-j\left[\beta(\omega_0)\, z - \omega_0\, t\right]\right\} \int_{-\infty}^{+\infty} \mathrm{V}(\Delta\omega)\ \exp\left[j\,\Delta\omega\left(t - \frac{z}{v_p}\right)\right] d\,\Delta\omega \qquad \text{[I-5]}$$

To account for dispersion, we Taylor expand $\beta(\omega)$ about the *center frequency* of the pulse -- *viz*.

$$\beta(\omega) = \beta(\omega_0) + \left.\frac{d\,\beta(\omega)}{d\omega}\right|_{\omega=\omega_0} (\omega - \omega_0) + \frac{1}{2!}\left.\frac{d^2\,\beta(\omega)}{d\omega^2}\right|_{\omega=\omega_0} (\omega - \omega_0)^2 + \ldots\ldots \qquad \text{[I-6a]}$$

Of course, such an expansion only makes sense if the line has relatively small dispersion. Fortunately, mathematical convenience here matches demands of practicality since highly dispersive lines are essentially useless in communication. For convenience and by convention, we may recast this equation as

$$\beta(\omega) \approx \beta(\omega_0) + v_g^{-1}\,(\omega - \omega_0) + b\,(\omega - \omega_0)^2 \qquad \text{[I-6b]}$$

where $v_g^{-1} = \left.\frac{d\,\beta(\omega)}{d\omega}\right|_{\omega=\omega_0}$ and $b = \frac{1}{2!}\left.\frac{d^2\,\beta(\omega)}{d\omega^2}\right|_{\omega=\omega_0}$. Substituting this expression into Equation [I-4], we obtain

$$\begin{aligned} \mathrm{V}(z,t) &= \exp\left\{-j\left[\beta(\omega_0)\, z - \omega_0\, t\right]\right\} \\ &\times \int_{-\infty}^{+\infty} \mathrm{V}(\omega - \omega_0)\ \exp\left\{-j\left[v_g^{-1}\,(\omega - \omega_0) + b\,(\omega - \omega_0)^2\right] z\right\} \exp\left[j\,(\omega - \omega_0)\, t\right] d\,(\omega - \omega_0) \end{aligned} \qquad \text{[I-7a]}$$

which may be written

$$\mathrm{V}(z,t) = \exp\left\{-j\left[\beta(\omega_0)\, z - \omega_0\, t\right]\right\}\ \mathrm{U}(z,t) \qquad \text{[I-7b]}$$

where

$$\mathrm{U}(z,t)=\int_{-\infty}^{+\infty}\mathrm{V}(\omega-\omega_0)\exp\left\{-j\left[v_g^{-1}(\omega-\omega_0)+b(\omega-\omega_0)^2\right]z\right\}\exp\left[j(\omega-\omega_0)t\right]d(\omega-\omega_0) \qquad \text{[I-8]}$$

which is the basis for both the **integral** and **differential equation** approaches to wave propagation on dispersive transmission lines.

II. PULSE DISPERSION DIFFERENTIAL EQUATION

Later we shall return to Equations [I-7] and [I-8] which represent the basis for the integral equation approach to wave propagation in dispersive media, but first we derive a differential equation which is the basis for the differential equation approach. To this end we differentiate the expression for $\mathrm{U}(z,t)$ -- *viz.*

$$\begin{aligned}\frac{\partial}{\partial z}\mathrm{U}(z,t)=\int_{-\infty}^{+\infty}&\mathrm{V}(\omega-\omega_0)\left\{-j\left[v_g^{-1}(\omega-\omega_0)+b(\omega-\omega_0)^2\right]\right\}\\ &\times\exp\left\{-j\left[v_g^{-1}(\omega-\omega_0)+b(\omega-\omega_0)^2\right]z\right\}\exp\left[j(\omega-\omega_0)t\right]d(\omega-\omega_0)\end{aligned} \qquad \text{[II-1]}$$

Therefore, we see that $\mathrm{U}(z,t)$ satisfies the following differential equation:

$$\frac{\partial}{\partial z}\mathrm{U}(z,t)+v_g^{-1}\frac{\partial}{\partial t}\mathrm{U}(z,t)=j\,b\,\frac{\partial^2}{\partial t^2}\mathrm{U}(z,t) \qquad \text{[II-2]}$$

For **minimal dispersion** -- *viz.* if $b=\frac{1}{2!}\left.\frac{d^2\beta(\omega)}{d\omega^2}\right|_{\omega=\omega_0}\approx 0$ -- then Equation [II-2] becomes

$$\frac{\partial}{\partial z}\mathrm{U}(z,t)+v_g^{-1}\frac{\partial}{\partial t}\mathrm{U}(z,t)=0 \qquad \text{[II-3]}$$

which is the basic wave equation for **minimally dispersive media** with the general solution

$$U(z,t) = U\left(z - v_g\, t\right) \qquad \text{[II-4]}$$

where $v_g = \left[\left. \frac{d\beta(\omega)}{d\omega} \right|_{\omega=\omega_0} \right]^{-1}$ is the **group velocity** of the pulse.

When limitation to **first order dispersion** is an adequate approximation -- *viz.* when $\left. \frac{1}{3!}\frac{d^3\beta(\omega)}{d\omega^3} \right|_{\omega=\omega_0} \approx 0$ -- the treatment of Equation [II-2] is facilitated by defining a new set of variables which transform us into a coordinate system which moves with the "group" -- *i.e.* what might be called the **surfer's coordinates**. Let

$$\tau = t - t_0 - v_g^{-1}\left(z - z_0\right) \text{ and } \varsigma = z - z_0 \qquad \text{[II-5]}$$

where $z_0 = v_g\, t_0$ so that

$$\begin{aligned} \frac{\partial}{\partial z} U(z,t) &= \frac{\partial}{\partial \varsigma} U(\varsigma,\tau) \frac{\partial \varsigma}{\partial z} + \frac{\partial}{\partial \tau} U(\varsigma,\tau) \frac{\partial \tau}{\partial z} \\ &= \frac{\partial}{\partial \varsigma} U(\varsigma,\tau)\,(1) + \frac{\partial}{\partial \tau} U(\varsigma,\tau) \left(-v_g^{-1}\right) \end{aligned} \qquad \text{[II-6a]}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} U(z,t) &= \frac{\partial}{\partial \varsigma} U(\varsigma,\tau) \frac{\partial \varsigma}{\partial t} + \frac{\partial}{\partial \tau} U(\varsigma,\tau) \frac{\partial \tau}{\partial t} \\ &= \frac{\partial}{\partial \varsigma} U(\varsigma,\tau)\,(0) + \frac{\partial}{\partial \tau} U(\varsigma,\tau)\,(1) \end{aligned} . \qquad \text{[II-6b]}$$

Therefore, Equation [II-2] becomes

$$b\,\frac{\partial^2}{\partial \tau^2} U(\varsigma,\tau) + j\,\frac{\partial}{\partial \varsigma} U(\varsigma,\tau) = 0 \qquad \text{[II-7]}$$

Amazingly, this pulse dispersion equation is the, so called, parabolic equation that we saw earlier in connection with beam propagation -- ***viz*****. the paraxial wave propagation equation.**

Solution of Pulse Dispersion Equation

For convenience, we restate here Equation [II-7} the first-order pulse dispersion equation -- *viz*.

$$b \frac{\partial^2}{\partial \tau^2} U(\varsigma, \tau) + j \frac{\partial}{\partial \varsigma} U(\varsigma, \tau) = 0 \ .$$

Let us write a Fourier transform for this modulation in terms of "surfer time" -- *i.e.*

$$U(\varsigma, \tau) = \int_{-\infty}^{+\infty} U(\varsigma, \omega) \exp(j\,\omega\,\tau)\, d\omega \qquad \text{[II-8a]}$$

where

$$U(\varsigma, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(\varsigma, \tau) \exp(-j\,\omega\tau)\, d\tau \ . \qquad \text{[II-8b]}$$

Thus, the pulse dispersion equation reduces to an ordinary differential equation -- *viz*.

$$j \frac{d}{d\varsigma} U(\varsigma, \omega) = b\,\omega^2\ U(\varsigma, \omega) \qquad \text{[II-9]}$$

for the Fourier transform and thus we have the simple solution

$$U(\varsigma, \omega) = U(0, \omega) \exp(-j\ b\,\omega^2\,\varsigma) \ . \qquad \text{[II-10]}$$

Thus, we see that the dispersion changes the phase of **each spectral component** of the pulse by an amount that depends on the frequency and the propagated distance. The general solution may be written as

$$\mathrm{U}(\varsigma,\tau) = \int_{-\infty}^{+\infty} \mathrm{U}(0,\omega)\,\exp\left[j\left(\omega\tau - b\,\omega^2\,\varsigma\right)\right] d\omega \qquad \text{[II-11a]}$$

where

$$\mathrm{U}(0,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{U}(0, t-t_0)\,\exp\left[-j\,\omega\,(t-t_0)\right] d\,(t-t_0)\ . \qquad \text{[II-11b]}$$

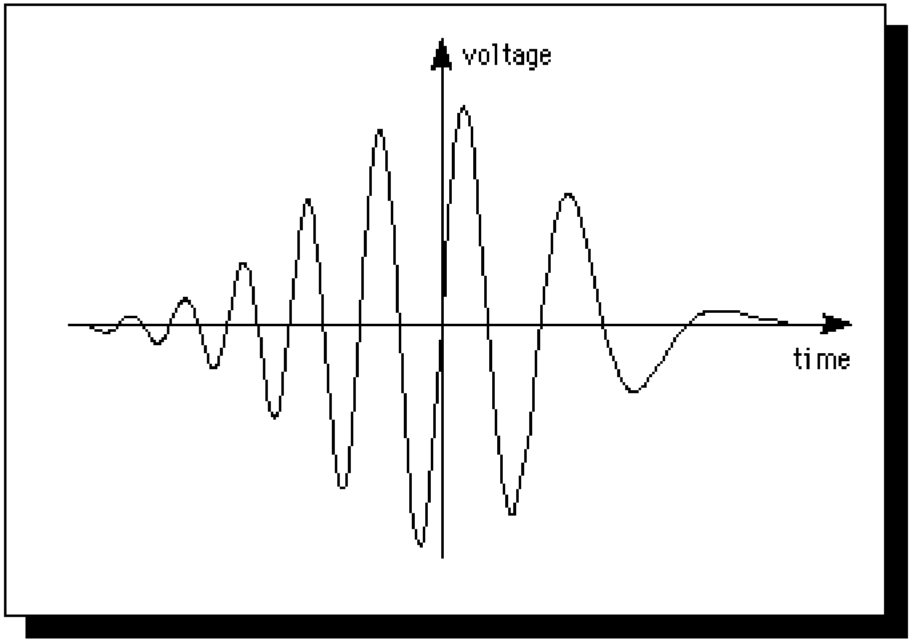
Let us suppose that we have a "chirped" Gaussian at $\varsigma = 0$ -- *i.e.*

$$\mathrm{U}(0, t-t_0) = \exp\left[-\left(\frac{1-j\,C}{2}\right)\left(\frac{t-t_0}{\tau_0}\right)^2\right] \qquad \text{[II-12a]}$$

so that -- see Equations [I-3a] and [I-3b] above --

$$\mathrm{U}(0,\omega) = \sqrt{\frac{\tau_0^2}{2\pi\,(1-j\,C)}}\ \exp\left[-\frac{\tau_0^2\,\omega^2}{2\,(1-jC)}\right] \qquad \text{[II-12b]}$$

where $C > 0$ characterizes an "up-chirp" and $C < 0$ a "down-chirp" pulse.



A Gaussian pulse with a frequency "down-chirp"

Inverting the transform, we see that

$$\mathrm{U}(\varsigma,\tau)=\left[\frac{\tau_0^2}{\tau_0^2+j\,2\,b\,\varsigma(1-j\,C)}\right]\exp\left\{-\frac{(1-j\,C)\,\tau^2}{2\left[\tau_0^2+j\,2\,b\,\varsigma(1-j\,C)\right]}\right\}\quad\text{[II-13a]}$$

or rationalizing

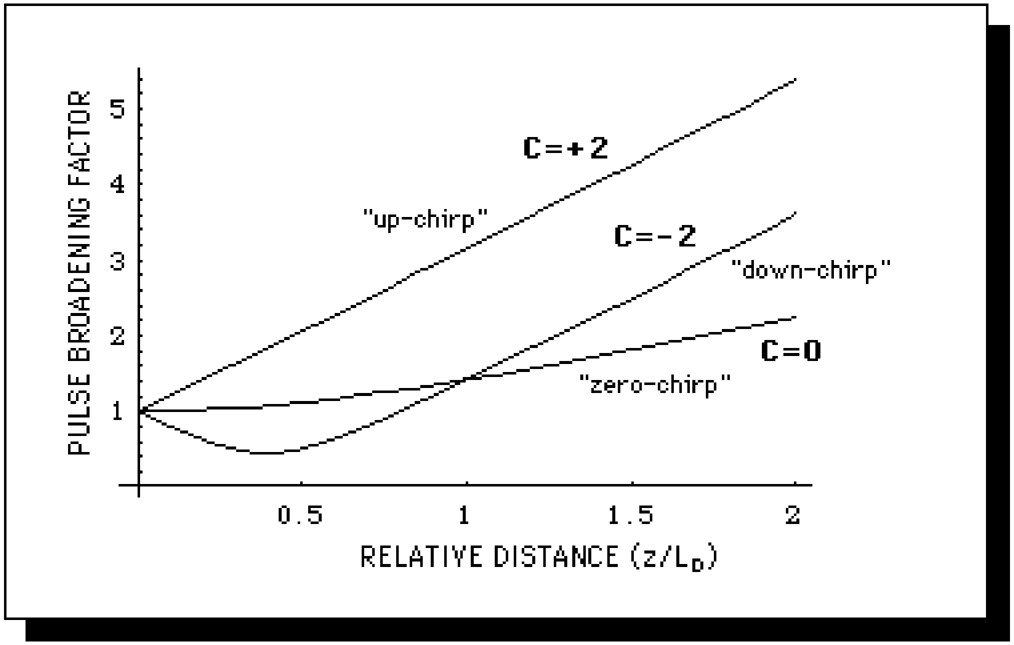
$$U(\zeta,\tau) = \left[\frac{\tau_0^2}{\tau_0^2 + j\,2\,b\left(1 - j\,C\right)}\right]\exp\left\{-\frac{\tau^2}{2\,\tau_0^2\left[\left(1+\frac{2\,b\,C\,\zeta}{\tau_0^2}\right)^2 + \left(1+\frac{2\,b\,\zeta}{\tau_0^2}\right)^2\right]}\right\}$$

$$\times\exp\left\{\frac{j\left[C + \frac{2\,b\,\zeta}{\tau_0^2}\left(1+C^2\right)\right]\tau^2}{2\,\tau_0^2\left[\left(1+\frac{2\,b\,C\,\zeta}{\tau_0^2}\right)^2 + \left(1+\frac{2\,b\,\zeta}{\tau_0^2}\right)^2\right]}\right\} \qquad \text{[II-13b]}$$

Hence, the **pulse width broadening factor** at a given position z is given by

$$\frac{\tau(\zeta)}{\tau_0} = \sqrt{\left(1+\frac{2\,b\,C\,\zeta}{\tau_0^2}\right)^2 + \left(\frac{2\,b\,\zeta}{\tau_0^2}\right)^2} = \sqrt{\left(1+\frac{C\,\zeta}{L_D}\right)^2 + \left(\frac{\zeta}{L_D}\right)^2} \qquad \text{[II-14]}$$

where $L_D = \tau_0^2/2\,b$.



The spatial evolution of the pulse width of chirped Gaussian pulse (For "normal dispersion" -- *i.e.* $\frac{\partial^2\beta}{\partial\omega^2} > 0$)

III. PULSE DISPERSION INTEGRAL EQUATION

We now return to the integral equation approach to dispersive wave propagation. Recall that from Equation [I-8]

$$\mathrm{U}(z,t) = \int_{-\infty}^{+\infty} \mathrm{V}(\omega-\omega_0)\exp\left\{-j\left[v_g^{-1}(\omega-\omega_0) + b\,(\omega-\omega_0)^2\right]z\right\}\exp\left[j\,(\omega-\omega_0)\,t\right]d\,(\omega-\omega_0)\ ,$$

but from Equation [I-2] we know that

$$\mathrm{V}(\omega) = \frac{1}{2\pi}\exp\left[j\,\beta(\omega)\,z_{ref}\right]\int_{-\infty}^{+\infty}\mathrm{V}(z_{ref},t)\exp\left[-j\,\omega\,t\right]dt$$

or that

$$\mathrm{V}(\omega-\omega_0) = \frac{1}{2\pi}\exp\left\{ j\left[v_g^{-1}(\omega-\omega_0) + b(\omega-\omega_0)^2\right] z_{ref}\right\} \int_{-\infty}^{+\infty} \mathrm{U}(z_{ref}, t)\exp\left[-j(\omega-\omega_0)\,t\right] dt \ . \tag{III-1}$$

Therefore,

$$\begin{aligned}\mathrm{U}(z, t) = \int_{-\infty}^{+\infty} \Bigg\{ & \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{U}(z_{ref}, t')\exp\left[-j(\omega-\omega_0)\,t'\right] dt' \\ & \times \exp\left\{-j\left[v_g^{-1}(\omega-\omega_0) + b(\omega-\omega_0)^2\right](z-z_{ref})\right\} \exp\left[j(\omega-\omega_0)\,t\right] d(\omega-\omega_0)\Bigg\} \ .\end{aligned} \tag{III-2}$$

If we reverse the order of integration, we obtain

$$\begin{aligned}\mathrm{U}(z, t) = & \int_{-\infty}^{+\infty} \mathrm{U}(z_{ref}, t')\ dt' \\ & \times \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{-j\left[v_g^{-1}(\omega-\omega_0) + b(\omega-\omega_0)^2\right](z-z_{ref})\right\} \exp\left[j(\omega-\omega_0)(t-t')\right] d(\omega-\omega_0) \ .\end{aligned} \tag{III-3a}$$

or

$$\mathrm{U}(z, t) = \int_{-\infty}^{+\infty} \mathrm{U}(z_{ref}, t')\left[\ \text{"integral"}\ \right] dt' \ . \tag{III-3b}$$

where

$$\text{"integral"} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{+j\,\omega'\left[(t-t') - v_g^{-1}(z-z_{ref})\right]\right\} \exp\left[-j\,b\,\omega'^2(z-z_{ref})\right] d\omega' \ . \tag{III-4a}$$

To make use of the **surfer coordinates** defined earlier, we write

$$\text{"integral"} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{+j\,\omega'\left[\tau+\left(t_0-t'\right)\right]\right\}\,\exp\left(-j\,b\,\omega'^2\,\varsigma\right)\,d\omega' \;. \qquad \text{[III-4b]}$$

where $\tau = t - t_0 - v_g^{-1}\left(z-z_0\right)$, $\varsigma = z - z_0$, and $z_0 = z_{ref}$. We evaluate this integral by completing the square -- *viz*.

$$\begin{aligned}\text{"integral"} &= \frac{1}{2\pi}\exp\left\{+j\,b\,\varsigma\left[\frac{\tau+\left(t_0-t'\right)}{2\,b\,\varsigma}\right]^2\right\}\int_{-\infty}^{+\infty}\exp\left\{-j\,b\,\varsigma\left[\omega'^2-\frac{\tau+\left(t_0-t'\right)}{b\,\varsigma}\,\omega'+\left(\frac{\tau+\left(t_0-t'\right)}{2\,b\,\varsigma}\right)^2\right]\right\}d\omega' \\ &= \frac{1}{2\pi}\exp\left\{+j\,b\,\varsigma\left[\frac{\tau+\left(t_0-t'\right)}{2\,b\,\varsigma}\right]^2\right\}\int_{-\infty}^{+\infty}\exp\left\{-j\,b\,\varsigma\left[\omega'-\frac{\tau+\left(t_0-t'\right)}{2\,b\,\varsigma}\right]^2\right\}d\omega'\end{aligned} \;. \qquad \text{[III-5]}$$

and, then, using the standard form

$$\int_{-\infty}^{+\infty}\exp\left[-a^2\,u^2\right]\;du = \sqrt{\pi}/a \qquad \text{[III-6]}$$

we obtain

$$\text{"integral"} = \frac{1}{2\pi}\sqrt{\frac{\pi}{j\,b\,\varsigma}}\,\exp\left\{+j\,b\,\varsigma\left[\frac{\tau+\left(t_0-t'\right)}{2\,b\,\varsigma}\right]^2\right\} \qquad \text{[III-7]}$$

Finally, we see that

$$\mathrm{U}(z,\,t) = \sqrt{\frac{1}{j\,4\pi\,b\,\varsigma}}\;\int_{-\infty}^{+\infty}\mathrm{U}\left(z_{ref},t'\right)\exp\left\{+j\,b\,\varsigma\left[\frac{\left(t'-t_0\right)-\tau}{2\,b\,\varsigma}\right]^2\right\}d\,t' \;. \qquad \text{[III-8a]}$$

or

$$\mathrm{U}(z, t) = \int_{-\infty}^{+\infty} \mathrm{U}(z_{ref}, t')\, h(t' - \tau - t_0)\, dt' \qquad \text{[III-8b]}$$

where

$$h(t' - \tau - t_0) = \sqrt{\frac{1}{j\, 4\pi\, b\, \varsigma}}\quad \exp\left\{ +j\, b\, \varsigma \left[\frac{(t' - t_0) - \tau}{2\, b\, \varsigma} \right]^2 \right\}. \qquad \text{[III-9]}$$

Thus, the "impulse response" -- *i.e.* the response for a sharp spike at $t' = t_0$ -- is given by

$$\mathrm{U}(z, t) = \sqrt{\frac{1}{j\, 4\pi\, b\, \varsigma}}\quad \exp\left(+j\, \frac{\tau^2}{4\, b\, \varsigma} \right). \qquad \text{[III-10]}$$