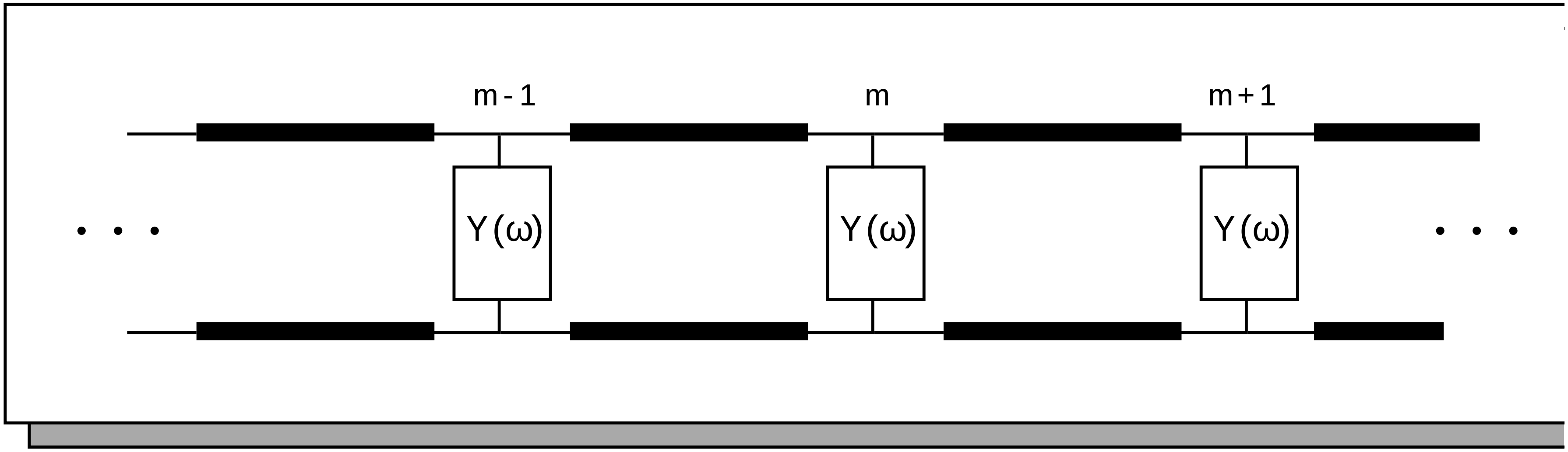
Consider a lossless transmission with wavenumber $\beta(\omega)$ and characteristic impedance $Z_c(\omega)$ which is periodically *loaded* with a shunt admittance $Y(\omega)$.




From transmission line theory, we know that we can write the spatial dependence of the voltage and current phasors as

$$V(z,\omega) = V_+(\omega)\,\exp\left[-j\,\beta(\omega) z\right] + V_-(\omega)\,\exp\left[+j\,\beta(\omega) z\right] \qquad \text{[ I-1a ]}$$

$$I(z,\omega) = Z_c^{-1}(\omega)\left\{V_+(\omega)\,\exp\left[-j\,\beta(\omega) z\right] - V_-(\omega)\,\exp\left[+j\,\beta(\omega) z\right]\right\} . \qquad \text{[ I-1a ]}$$

The propagation from one shunt admittance to the next is most conveniently expressed in matrix form as

$$\begin{bmatrix} V(d,\omega) \\ I(d,\omega) \end{bmatrix} = \begin{bmatrix} \cos\left[\beta(\omega) d\right] & -j\,Z_c(\omega)\sin\left[\beta(\omega) d\right] \\ -j\,Z_c^{-1}(\omega)\sin\left[\beta(\omega) d\right] & \cos\left[\beta(\omega) d\right] \end{bmatrix} \begin{bmatrix} V(0,\omega) \\ I(0,\omega) \end{bmatrix} \qquad \text{[ I-2 ]}$$

where $d$ is the periodic spacing of the admittances. Thus, if $V_m$ and $I_m$ are, respectively,

$$\ldots\; I_{m+1}(\omega) \;\ldots\; -j\, Z_c^{-1}(\omega)\, \sin[\beta(\omega)\, d] \qquad \cos[\beta(\omega)\, d] \;\ldots\; -Y(\omega) \quad 1 \;\ldots\; I_m(\omega) \qquad [\text{ I-3a }]$$

$$\begin{bmatrix} V_{m+1}(\omega) \\ I_{m+1}(\omega) \end{bmatrix} = \begin{bmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{bmatrix} \begin{bmatrix} V_m(\omega) \\ I_m(\omega) \end{bmatrix} \qquad [\text{ I-3b }]$$

where

$$\begin{bmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{bmatrix} = \begin{bmatrix} \cos[\beta(\omega)\, d] + j\; Y(\omega)\; Z_c(\omega) \sin[\beta(\omega)\, d] & -j\, Z_c(\omega) \sin[\beta(\omega)\, d] \\ -j\, Z_c^{-1}(\omega) \sin[\beta(\omega)\, d] - Y(\omega) \cos[\beta(\omega)\, d] & \cos[\beta(\omega)\, d] \end{bmatrix} . \qquad [\text{ I-4 }]$$

That is to say, using the **ABCD** matrix defined in this way, we may write

$$V_{m+1} = A V_m + B\, I_m \qquad [\text{ I-5a }]$$

and

$$I_{m+1} = C\, V_m + D\, I_m \qquad [\text{ I-5b }]$$

From Equation [ I-5a ] we may write

$$I_m = \frac{V_{m+1} - A\, V_m}{B} \quad \text{and} \quad I_{m+1} = \frac{V_{m+2} - A\, V_{m+1}}{B} \qquad [\text{ I-6 }]$$

which when substituted into Equation [ I-5b ] yields

$$V_{m+2} - [A + D]\, V_{m+1} + [A\,D - BC]\, V_m = 0 \qquad [\text{ I-7a }]$$

We can show that $[A\,D - B\,C] = 1$ so that Equation [ I-7b ] reduces immediately to [1]

---

[1] Since $[A\,D - B\,C] = 1$ , we have the inverse relationship

and from Equation [ I-4 ] we see that

$$A + D = 2\cos\beta d + j\,Y\,Z_c \sin\beta d \ . \qquad [\text{ I-8 }]$$

Drawing on past experience, we look for solutions of the form $V_m = V_0 \exp(j\,m\varphi)$ which are possible if and only if

$$\exp[j\,\varphi(\omega)] + \exp[-j\,\varphi(\omega)] = 2\cos[\varphi(\omega)] = A(\omega) + D(\omega) \qquad [\text{ I-9a }]$$

or

$$\cos\varphi(\omega) = \cos[\beta(\omega)\,d] + j\,\frac{1}{2}\,Y(\omega)\,Z_c(\omega)\sin[\beta(\omega)\,d] \ . \qquad [\text{ I-9b }]$$

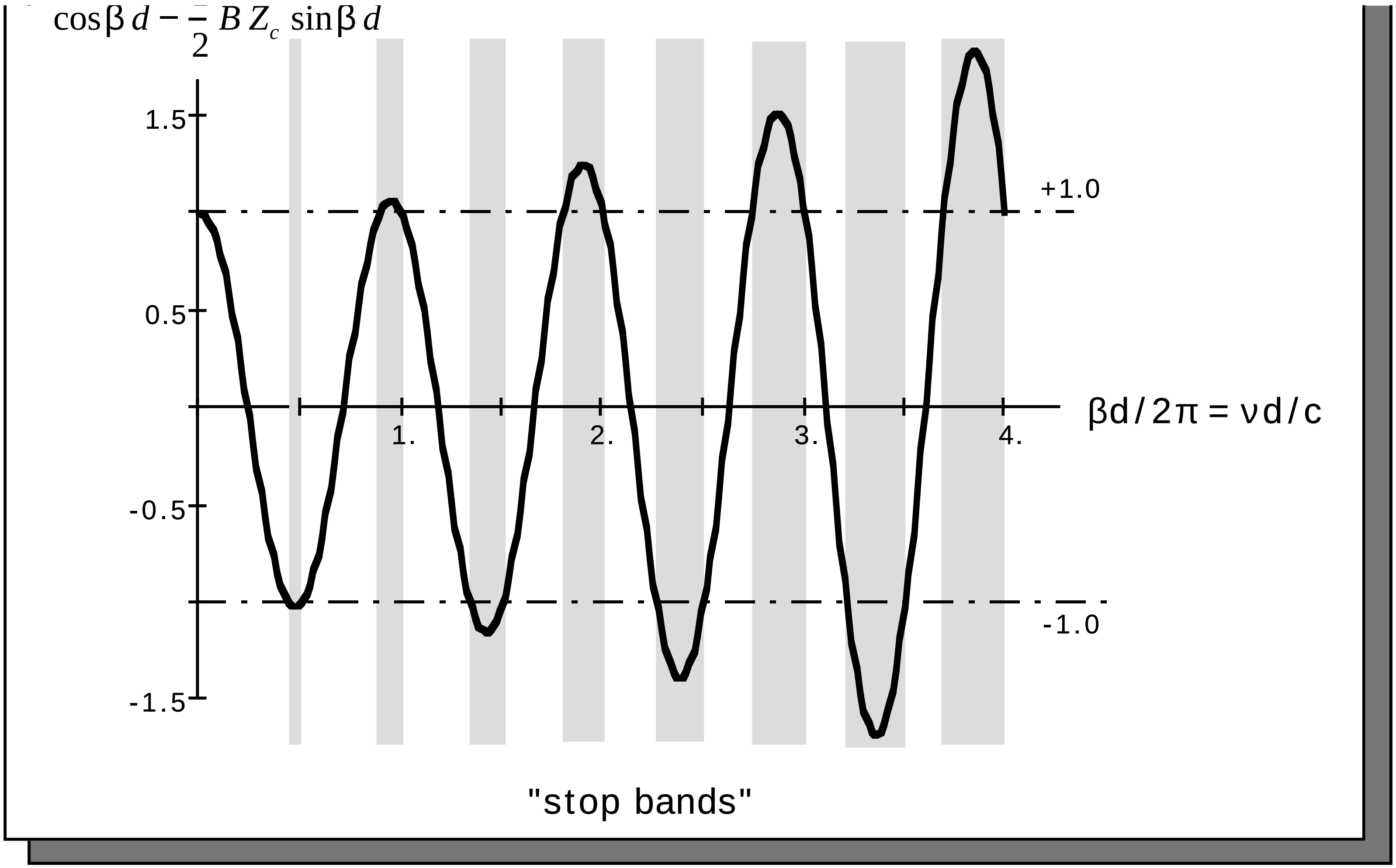
For purely reactive loading $Y(\omega) = j\,B(\omega)$ so that

$$\cos\varphi(\omega) = \cos[\beta(\omega)\,d] - \frac{1}{2}\,B(\omega)\,Z_c(\omega)\sin[\beta(\omega)\,d] \ . \qquad [\text{ I-10 }]$$

For wave propagation through the periodic system, $\varphi$ must be real or we must have

$$|\cos\varphi| = \left|\cos[\beta(\omega)\,d] - \frac{1}{2}\,B(\omega)\,Z_c(\omega)\sin[\beta(\omega)\,d]\right| \leq 1 \ . \qquad [\text{ I-11 }]$$

**Thus, there may be frequencies at which propagation is not possible!**




**Condition embodied in Equation [ I-11 ] for a capacitively loaded line.**

We can get further insight into wave propagation in a periodic system by examining one particularly simple case. To that end, we first re-expressing Equation [ I-11 ] in the form

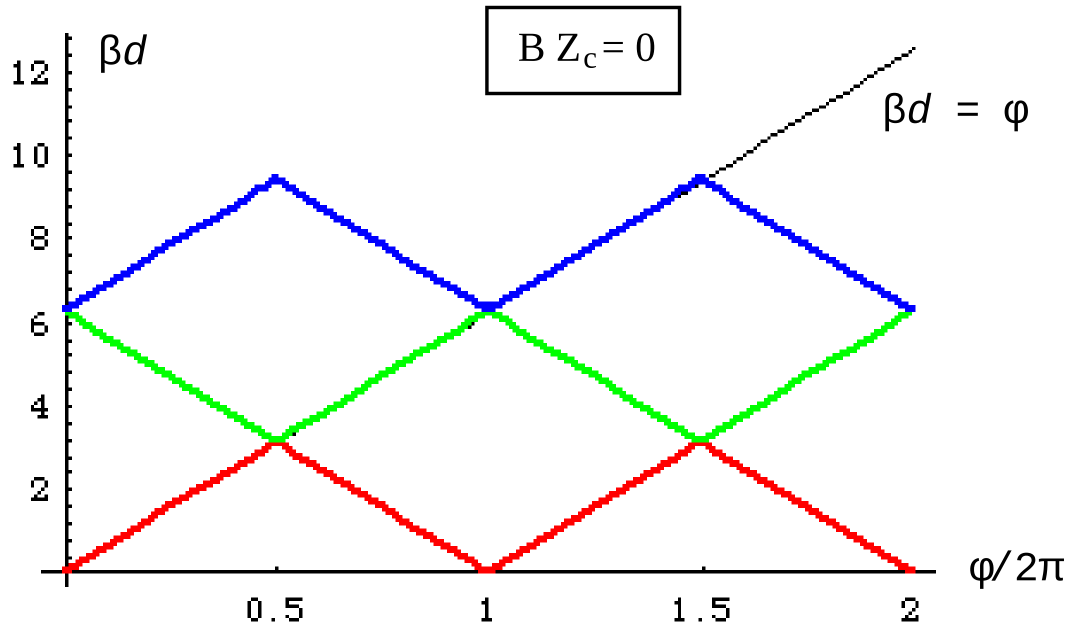
$$\cos[\varphi(\omega)] = \sec[\theta(\omega)] \cos[\beta(\omega) d + \theta(\omega)] \qquad \text{[ I-12 ]}$$

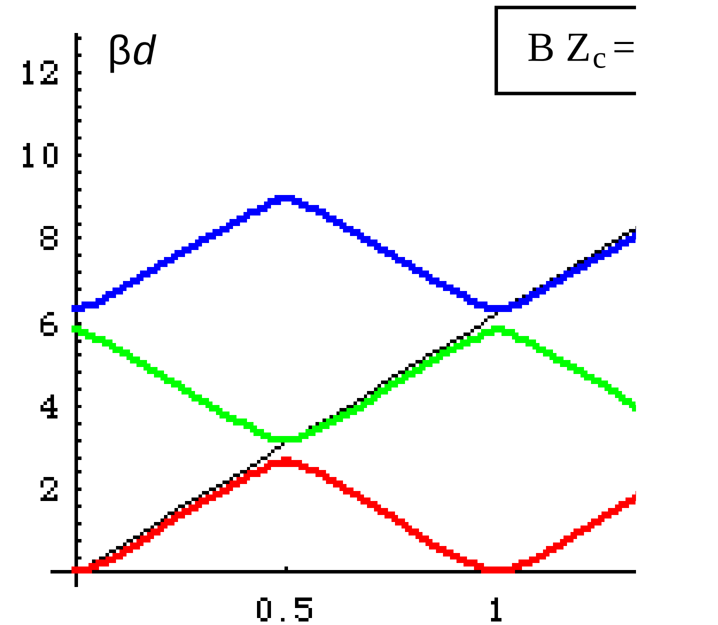
where $\tan[\theta(\omega)] \equiv B(\omega)\, Z_c(\omega)/2$ and $\sec[\theta(\omega)] = \sqrt{1 + (B(\omega)\, Z_c(\omega)/2)^2}$ .

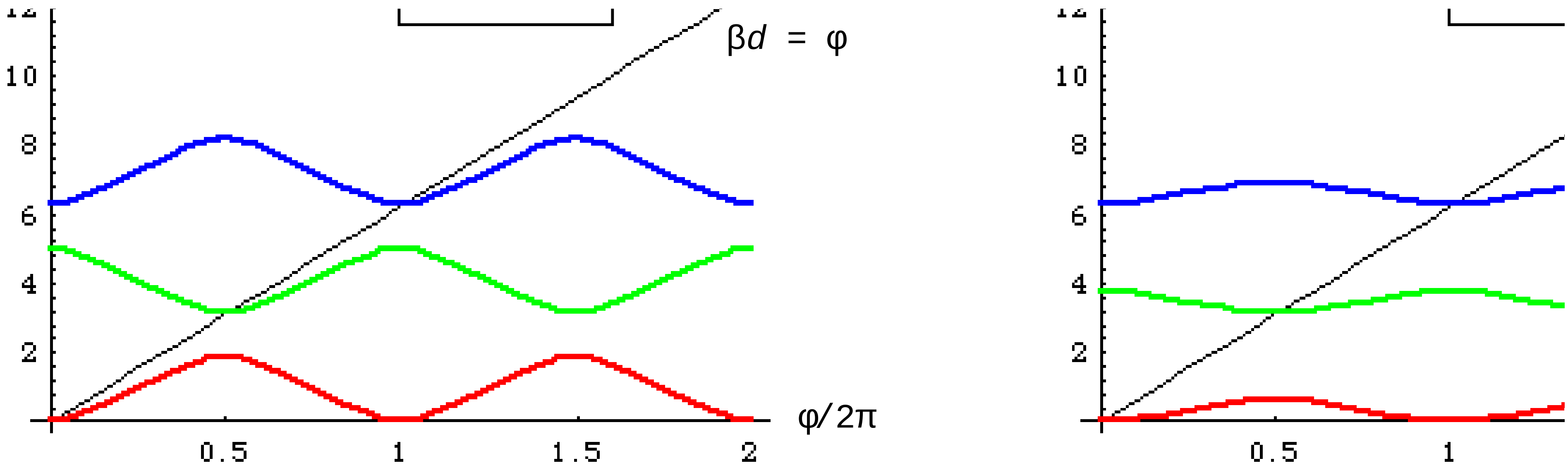
write

$$\beta d = \cos^{-1}\left[\cos\theta \cos(\varphi + \pi n)\right] - \theta + \pi n \ . \qquad [\text{ I-13 }]$$

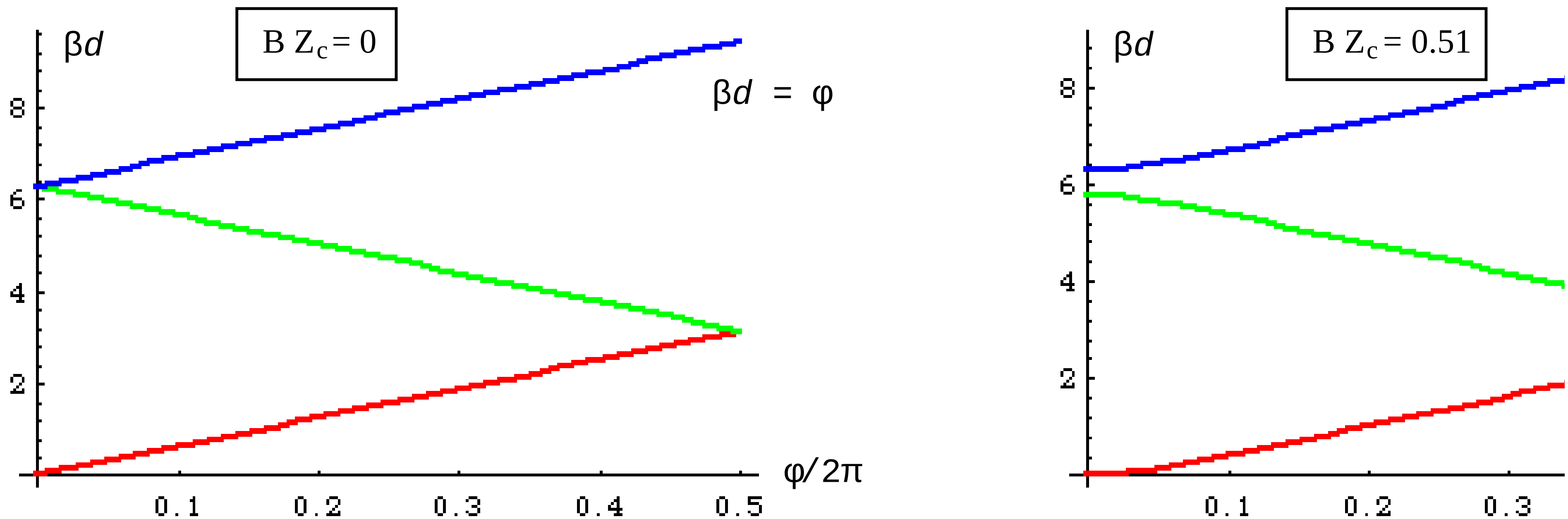
where *n* is some integer. The development of a ***pass band*** or ***forbidden gap*** with increasing values of the $B(\omega)\,Z_c(\omega)$ product is clearly shown in the follow set of dispersion relationships -- *i.e.* Equation [ I-13 ]:

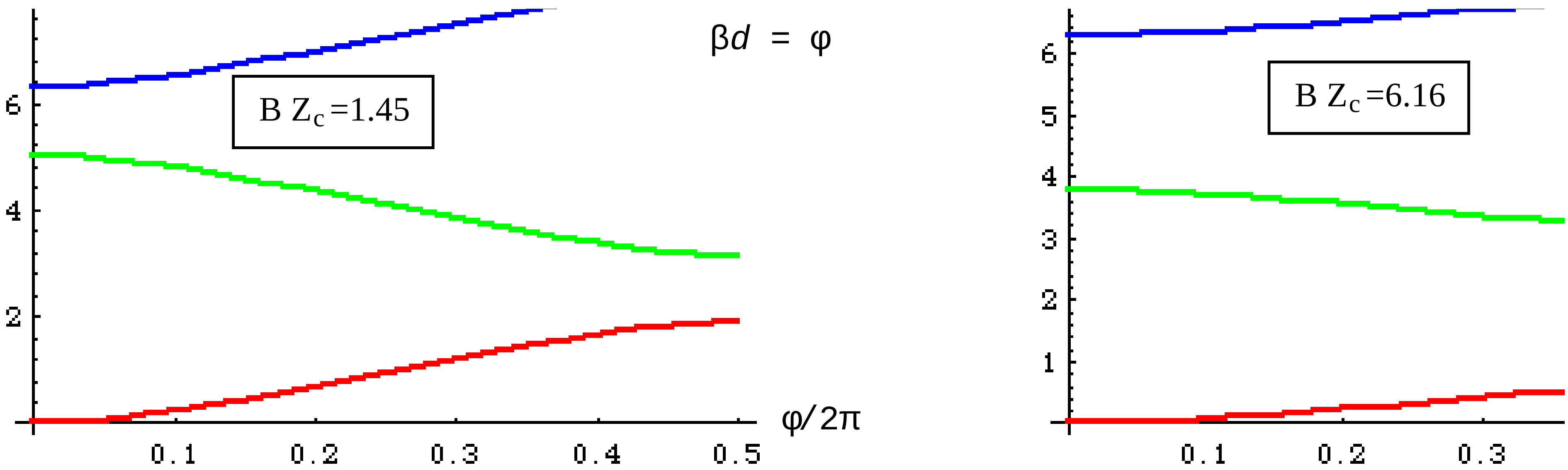





Since all the information is contained in the region $\varphi \leq \pi$ , the dispersion relationships are usually plotted in this so-called ***reduced zone*** -- *viz.*




βd = φ
B Zc =1.45
6
4
2
φ/2π
0.1
0.2
0.3
0.4
0.5
B Zc =6.16
6
5
4
3
2
1
0.1
0.2
0.3