Guided Wave Formulation of Maxwell's Equations

I. General Theory:

Recapitulation -- frequency domain formulation of the macroscopic Maxwell equations in a source-free region:

\[
\text{curl } \vec{E}(\vec{r}, \omega) = -j\omega \mu \vec{H}(\vec{r}, \omega) \quad \text{[I-1a]}
\]

\[
\text{curl } \vec{H}(\vec{r}, \omega) = +j\omega \varepsilon \vec{E}(\vec{r}, \omega) \quad \text{[I-1b]}
\]

\[
\text{div } \varepsilon \vec{E}(\vec{r}, \omega) = 0 \quad \text{[I-1c]}
\]

\[
\text{div } \mu \vec{H}(\vec{r}, \omega) = 0 \quad \text{[I-1d]}
\]

Important Review:
Let us once again derive the electromagnetic wave equations (or more precisely, the electromagnetic Helmholtz equations) by starting with Eq. [I-1a]. as we have shown many times, the trick is to operate on this equation with the \text{curl} operator -- \text{viz.}

\[
\text{curl curl } \vec{E}(\vec{r}, \omega) = -j\omega \text{curl} \left[ \mu \vec{H}(\vec{r}, \omega) \right] = -j\omega \mu \left[ +j\omega \varepsilon \vec{E}(\vec{r}, \omega) \right] \quad \text{[I-2]}
\]

and then use the "abc = bac - cab" rule\textsuperscript{†} - \text{i.e.}

\[
\text{grad} \left[ \text{div } \vec{E}(\vec{r}, \omega) \right] - \left[ \text{div grad} \vec{E}(\vec{r}, \omega) \right] = \omega^2 \mu \varepsilon \vec{E}(\vec{r}, \omega) \quad \text{[I-3]}
\]

Finally, we make use of Eq. [I-1c] and obtain

\[
\left[ \text{div grad} \vec{E}(\vec{r}, \omega) + k^2 \vec{E}(\vec{r}, \omega) = \nabla^2 \vec{E}(\vec{r}, \omega) + k^2 \vec{E}(\vec{r}, \omega) = 0 \quad \text{[I-4a]}
\]

where \( k^2 = \omega^2 \mu \varepsilon \).

Similarly, starting from Eq. [I-1b] we obtain

\[\text{That is} \quad \vec{a} \times \left( \vec{b} \times \vec{c} \right) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})\]
These are the Helmholtz equations which are the appropriate starting point for the analysis of many problems in electromagnetic propagation. We have already used them to study plane wave propagation in uniform dielectrics.

**Treatment of Guided Waves:**

While the Helmholtz approach is a valuable starting point in many instances, we take a slightly different tack in analyzing guided wave problems. The important point of departure is that we assume the waves are propagating in the $z$-direction along a uniform guided wave structure and, thus, we may write in all differential operators in terms of transverse and longitudinal components by taking

$$\frac{\partial}{\partial z} \{\text{any field variable}\} = \gamma \{\text{any field variable}\}$$  \hspace{1cm} \text{[I-5a]}

so that

$$\vec{\nabla} \{\text{any field variable}\} = \vec{\nabla}_t \{\text{any field variable}\} + \hat{z} \gamma \{\text{any field variable}\}$$  \hspace{1cm} \text{[I-5b]}

where, for example and in particular,

$$\vec{\nabla}_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}.$$  \hspace{1cm} \text{[I-6]}

Furthermore, it is also useful to resolve the fields into transverse and longitudinal components -- *viz.*

$$\vec{E}(\vec{r}, \omega) = \vec{E}_t(\vec{r}, \omega) + \hat{z} \vec{E}_z(\vec{r}, \omega)$$  \hspace{1cm} \text{[I-7a]}

$$\vec{H}(\vec{r}, \omega) = \vec{H}_t(\vec{r}, \omega) + \hat{z} \vec{H}_z(\vec{r}, \omega)$$  \hspace{1cm} \text{[I-7b]}

Thus, we may rewrite Maxwell’s equations and resolve them into transverse and longitudinal components -- *viz.* Eq. [I-1a] becomes

$$\left\{\vec{\nabla}_t + \hat{z} \gamma\right\} \times \left\{\vec{E}_t(\vec{r}, \omega) + \hat{z} \vec{E}_z(\vec{r}, \omega)\right\} = -j \omega \mu \left\{\vec{H}_t(\vec{r}, \omega) + \hat{z} \vec{H}_z(\vec{r}, \omega)\right\}$$  \hspace{1cm} \text{[I-8a]}

which has transverse components
\[ \gamma \hat{z} \times \vec{E}_{tr}(\vec{r},\omega) - \hat{z} \times \vec{E}_{tr}(\vec{r},\omega) = -j\omega \mu \vec{H}_{tr}(\vec{r},\omega) \]  \[ \text{[I-8b]} \]
and longitudinal component
\[ \vec{\nabla}_{tr} \times \vec{E}_{tr}(\vec{r},\omega) = -j\omega \mu \vec{H}_{z}(\vec{r},\omega) \]  \[ \text{[I-8c]} \]
and Eq. [ I-1a ] becomes
\[ \left\{ \vec{\nabla}_{tr} + \hat{z} \gamma \right\} \times \left\{ \vec{H}_{tr}(\vec{r},\omega) + \hat{z} \vec{E}_{z}(\vec{r},\omega) \right\} = +j\omega \varepsilon \left\{ \vec{E}_{tr}(\vec{r},\omega) + \hat{z} \vec{E}_{z}(\vec{r},\omega) \right\} \]  \[ \text{[I-9a]} \]
which has transverse components
\[ \gamma \hat{z} \times \vec{H}_{tr}(\vec{r},\omega) - \hat{z} \times \vec{H}_{z}(\vec{r},\omega) = +j\omega \varepsilon \vec{E}_{tr}(\vec{r},\omega) \]  \[ \text{[I-9b]} \]
and longitudinal component
\[ \vec{\nabla}_{tr} \times \vec{H}_{tr}(\vec{r},\omega) = +j\omega \varepsilon \vec{E}_{z}(\vec{r},\omega) \]  \[ \text{[I-9c]} \]
Now multiply the first transverse equation -- i.e. Eq. [ I-8b ] -- by \( j\omega \varepsilon \)
\[ \gamma \hat{z} \times \left\{ j\omega \varepsilon \vec{E}_{tr}(\vec{r},\omega) \right\} - j\omega \varepsilon \hat{z} \times \vec{E}_{tr}(\vec{r},\omega) = \omega^2 \varepsilon \mu \vec{H}_{tr}(\vec{r},\omega) \]
and substitute from the second transverse equation -- i.e. Eq. [ I-9b ]
\[ \gamma \hat{z} \times \left\{ \gamma \hat{z} \times \vec{H}_{tr}(\vec{r},\omega) - \hat{z} \times \vec{H}_{z}(\vec{r},\omega) \right\} - j\omega \varepsilon \hat{z} \times \vec{E}_{tr}(\vec{r},\omega) = \omega^2 \varepsilon \mu \vec{H}_{tr}(\vec{r},\omega) \]
Finally, using the "abc = bac - cab" rule we find
\[ \left\{ k^2 + \gamma^2 \right\} \vec{H}_{tr}(\vec{r},\omega) = \gamma \vec{\nabla}_{tr} \vec{H}_{z}(\vec{r},\omega) - j\omega \varepsilon \hat{z} \times \vec{E}_{z}(\vec{r},\omega) \]  \[ \text{[I-10a]} \]
Similarly, multiplying the second transverse equation -- i.e. Eq. [ I-9b ] -- by \( -j\omega \mu \)
\[ \hat{a} \times \left( \hat{b} \times \hat{c} \right) = \hat{b} (\hat{a} \cdot \hat{c}) - \hat{c} (\hat{a} \cdot \hat{b}) \]  \[ \text{† That is} \]
\[ \hat{a} \times \left( \hat{b} \times \hat{c} \right) = \hat{b} (\hat{a} \cdot \hat{c}) - \hat{c} (\hat{a} \cdot \hat{b}) \]
\[ \gamma \hat{z} \times \left\{ j \omega \varepsilon \tilde{E}_m(\hat{r}, \omega) \right\} - j \omega \varepsilon \hat{z} \times \tilde{E}_m(\hat{r}, \omega) = \omega^2 \varepsilon \mu \tilde{H}_m(\hat{r}, \omega) \]

and substituting from the **first transverse equation** -- *i.e.* Eq. [I-8b] -- we obtain

\[
\left\{ k^2 + \gamma^2 \right\} \tilde{E}_m(\hat{r}, \omega) = \gamma \tilde{\nabla} \times \tilde{E}_m(\hat{r}, \omega) + j \omega \mu \hat{z} \times \tilde{\nabla} \tilde{H}_m(\hat{r}, \omega) \quad \text{[I-10b]}
\]

**In summary**, these equations say: **If we know the longitudinal electric and magnetic fields, we know everything these is to know!** But how are we to know the longitudinal fields? Answer: The longitudinal fields must, of course, obey the Helmholtz equations, which may now be written in the form

\[
\nabla^2 \tilde{E}_m(\hat{r}, \omega) + \left\{ k^2 + \gamma^2 \right\} \tilde{E}_m(\hat{r}, \omega) = 0 \quad \text{[I-11a]}
\]

\[
\nabla^2 \tilde{H}_m(\hat{r}, \omega) + \left\{ k^2 + \gamma^2 \right\} \tilde{H}_m(\hat{r}, \omega) = 0 \quad \text{[I-11b]}
\]

where, for example and in particular,

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

\[
= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \quad \text{[I-12]}
\]

**II. TEM Waveguide Modes:**

The argument above is incomplete in the sense that one important case has been implicitly overlooked - *i.e.* the case when both longitudinal fields are zero. When does this happen? Answer: when

\[
k^2 + \gamma^2 = 0 \quad \text{or} \quad \gamma = \pm j k \quad \text{[II-1]}
\]

Therefore, **TEM waves always propagate at the speed of light in the dielectric** and satisfy the equations

\[
\nabla^2 \tilde{E}_m(\hat{r}, \omega) = 0 \quad \text{[II-2a]}
\]
An important particular example of such TEM waveguide modes are those derived from the logarithmic potential \(-\text{viz.}\)
\[
\phi(\bar{r},\omega) = \frac{A}{2} \ln\left[(x + a)^2 + (y + b)^2\right] \quad \text{[II-3]}
\]
which includes the TEM modes on coaxial and two-wire lines.

III. Conducting Wall Tube Waveguides:

For the nearly realizable case of perfectly conducting walls we have the simplest possible situation the longitudinal fields satisfy the Helmholtz equations -- \(-\text{viz.}\) Eqs \([\text{I-11}]\) -- in the internal dielectric region and the following the boundary conditions on the walls:

\[
E_z(\hat{r}_{\text{wall}},\omega) = 0 \quad \text{[III-1a]}
\]
\[
\hat{z} \times \vec{\nabla}_{\text{tr}} H_z(\hat{r}_{\text{wall}},\omega) = 0 \quad \text{[III-1b]}
\]

A. Parallel Plate Waveguides

TM-Modes (E-Waves) are derived from
\[
E_z^n(\bar{r},\omega) = A_m \sin\left[\frac{m\pi x}{d}\right] \exp[j\gamma_m z] \quad \text{[III-2a]}
\]
where \( k^2 + \gamma_m^2 = \left(\frac{m\pi}{d}\right)^2 \) and thus \( \gamma_m = j\beta_m = \sqrt{\left(\frac{m\pi}{d}\right)^2 - k^2} \).

TE-Modes (H-Waves) are derived from
\[
H_z^n(\bar{r},\omega) = B_n \cos\left[\frac{n\pi x}{d}\right] \exp[j\gamma_n z] \quad \text{[III-2b]}
\]
where \( k^2 + \gamma_n^2 = \left(\frac{n\pi}{d}\right)^2 \) and thus \( \gamma_n = j\beta_n = \sqrt{\left(\frac{n\pi}{d}\right)^2 - k^2} \).
Dispersion Curves for a Parallel Plate Waveguide

From Eqs. [I-10], we may write the complete transverse fields for a given order

\[
\tilde{E}_m^r(\mathbf{r}, \omega) = \left[ k^2 + \gamma_m^2 \right]^{-1} \left\{ \gamma_m \tilde{\nabla}_r \left[ A_m \sin \left( \frac{m\pi x}{d} \right) \right] + j \omega \mu \hat{z} \times \tilde{\nabla}_r \left[ B_m \cos \left( \frac{m\pi x}{d} \right) \right] \right\} \exp[\gamma_m z] \quad [\text{III-3a}]
\]

\[
\tilde{H}_m^r(\mathbf{r}, \omega) = \left[ k^2 + \gamma_m^2 \right]^{-1} \left\{ \gamma_m \tilde{\nabla}_r \left[ B_m \cos \left( \frac{m\pi x}{d} \right) \right] - j \omega \epsilon \hat{z} \times \tilde{\nabla}_r \left[ A_m \sin \left( \frac{m\pi x}{d} \right) \right] \right\} \exp[\gamma_m z] \quad [\text{III-3a}]
\]

which simplify to

\[
\tilde{E}_m^r(\mathbf{r}, \omega) = \left[ \frac{d}{m\pi} \right] \left\{ j \beta_m A_m \cos \left( \frac{m\pi x}{d} \right) \hat{x} - j \omega \mu B_m \sin \left( \frac{m\pi x}{d} \right) \hat{y} \right\} \exp[j\beta_m z] \quad [\text{III-4a}]
\]
\[ \tilde{H}_{tr}^m(\tilde{r}, \omega) = \left[ \frac{d}{m\pi} \right] \left\{ -j \beta_m B_m \sin \left( \frac{m\pi x}{d} \right) \hat{x} - j \omega \varepsilon A_m \cos \left( \frac{m\pi x}{d} \right) \hat{y} \right\} \exp[j \beta_m z] \]  

[III-4b]

**B. Rectangular Waveguides**

**TM-Modes (E-Waves)** are derived from

\[ E_x^{mn}(\tilde{r}, \omega) = A_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \exp[\gamma_{mn} z] \]  

[III-5a]

where \( k^2 + \gamma_m^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \)

and thus \( \gamma_{mn} = j \beta_{mn} = \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} - k^2 \).

**TE-Modes (H-Waves)** are derived from

\[ H_z^{mn}(\tilde{r}, \omega) = B_{mn} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) \exp[\gamma_{mn} z] \]  

[III-5b]

where, again, \( k^2 + \gamma_m^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \)
and \[ \gamma_{mn} = j \beta_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} \] [ III-6 ]

**Dispersion Curves for a Rectangular Waveguide**
(Most common or standard configuration where \( a = 2b \))