EXPECTED AND UNEXPECTED SOLUTIONS
TO THE STATIONARY ONE-DIMENSIONAL
NONLINEAR SCHRÖDINGER EQUATION

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We present all stationary solutions to the nonlinear Schrödinger equation in one
dimension for box and periodic boundary conditions. For both repulsive and attractive
nonlinearity we find expected and unexpected solutions. Expected solutions
are those that are in direct analogy with those of the linear Schrödinger equation
under the same boundary conditions. Unexpected solutions are those that have no
such analogy. We give a physical interpretation for the unexpected solutions. We
discuss the properties of all solution types and briefly relate them to experiments
on the dilute-gas Bose-Einstein condensates.

The nonlinear Schrödinger equation (NLSE) models many phenomena observed
in the recently created dilute gas Bose-Einstein condensates (BECs). In this context
it is also referred to as the Gross-Pitaevskii equation. The NLSE has applications in
the description of a variety of wave propagation phenomena. It is one of a few basic
equations upon which the modern theory of integrable nonlinear systems has been
founded. Most applications of the NLSE to BECs have dealt with ground-state
properties, but there is growing interest in the possibility of generating topological
excitations of a condensate, which may well be described by excited-state solutions
of the NLSE.

In this article we present such solutions for the cases of a one-dimensional NLSE
for repulsive and attractive nonlinearity, subject either to box or periodic boundary
conditions on a finite interval. This corresponds to a BEC with repulsive or attractive
atomic pair interactions, respectively. The excited-state solutions, which may
be written in closed analytic form, offer insight into the basic geometry of topological
excitations in simplified models. For example, the NLSE with box boundary
conditions on a finite interval is an idealization of a BEC in a high-aspect-ratio
atomic trap the case of periodic boundary conditions resembles a BEC confined in
a thin annular trap.

By scaling the coordinates and wavefunction normalization we can reduce the
general NLSE to the simple form:
Figure 1. Expected solutions: the NLSE analogue to the linear quantum mechanics particle-in-a-box and particle-on-a-ring real stationary solutions. In each case the first three modes in the box and the first mode on the ring are shown. (a)-(c): attractive cases; (d)-(f): repulsive case. (a) and (d) are the ground state in a box, (b) and (e) are the first excited states in the box and also the first real solution on the ring. The number of nodes in the box is 0, 1, 2, ... while on the ring it is 2, 4, 6, ..., just as in the analogous linear quantum mechanics solutions.

\[ \partial_{xx} \pm |u|^2 + V(x) \] \[ \mu u \] (1)

where \( u \) is the wavefunction, \( x \) is normalized to the unit interval, \( \mu \) is the chemical potential which is an eigenvalue, and all quantities are unitless. The \( \pm \) sign on the nonlinear term refers to the attractive and repulsive cases, respectively. The first term is the kinetic energy, the second the mean-field energy, and the third the trapping potential, which in the cases considered here will amount to a choice of boundary conditions.

We seek solutions of Eq. (1) subject to normalization and boundary conditions. These three conditions uniquely determine a sequence of stationary states. We shall not detail their mathematical form here, as this appears elsewhere \(^1\)\(^2\), but we note that all solutions may be written in terms of Jacobian elliptic functions.

Intuitively one expects that a nonlinear deviation from the known sinusoidal solutions of the linear particle-in-a-box and particle-on-a-ring problems should bring about a dumping together or a spreading out of the wavefunction between nodes in the attractive and repulsive cases, respectively. It is exactly this effect that is seen in the plots of the \textit{expected} solutions in Fig. 1. All of these solutions are real. On the ring they break symmetry. We also note that, as in the linear Schrödinger equation, there are complex, constant-amplitude, plane-wave solutions on the ring, and these too are expected.

The \textit{unexpected} solutions are nodeless, symmetry-breaking, stationary states on the ring. In the attractive case there are both real and complex solutions, shown in Fig. 2(a)-(c) and (d)-(f), respectively. In the repulsive case there are only complex ones, shown in Fig. 2(g)-(i).
Figure 2. Unexpected solutions: nodeless, symmetry-breaking, stationary states on the ring. None of these solutions have an analogue in the particle-on-a-ring problem in linear quantum mechanics. In plots (d)-(i) r is the amplitude and \( \phi \) is the phase, where \( u = r \exp(i\phi) \). Attractive case: (a) the ground state; (b) the nearly degenerate symmetric partner to the antisymmetric solution shown in Fig. 1(b); (c) the third mode of this real solution type; (d)-(f) 3, 4, and 5-peaked complex solutions. Repulsive case: (g)-(i) 1, 2, and 3 density-notch complex solutions. Interpretations of the unexpected solutions are given in the text.

The complex solutions have a monotonically increasing phase. Each has a degenerate partner in which the phase monotonically decreases but is otherwise the same. There is no analogue of these solutions in the particle-on-a-ring problem in linear quantum mechanics. The complex solutions for the attractive case, in particular, are an entirely new solution-type to the NLSE.

The complex solutions for the repulsive case are interpreted as density-notch solitons moving with speed \( c \) on the ring with an opposing momentum boost of the condensate of speed \(-c\), which results in a stationary state in the lab frame. Density-notch solitons have a speed between zero and the Bogoliubov sound speed, ranging from maximal to zero depth, respectively. Those not of maximal depth are called grey solitons, while those which are of maximal depth and therefore form a node are called dark. Fig. 1(d)-(f) shows the bounded version of a dark \( j \)-soliton train, where \( j \) is the number of nodes. Fig. 2(g)-(i) shows the bounded, quantized version of a grey \( j \)-soliton train, where \( j \) is the number of density minima.

At typical experimental trap sizes the single density-notch solution on the ring, as shown in Fig. 2(g), is the lowest energy symmetry-breaking excitation above the real, constant-amplitude ground state. When any of these stationary excited
states is perturbed it gives rise to soliton-type motions. Recent reports suggest that such motions can be induced in repulsive BECs by optical manipulation of the condensate phase.

All attractive symmetry-breaking solution types, i.e., the expected anti-symmetric ones shown in Fig. 1(a)-(c), the unexpected real ones shown in Fig. 2(a)-(c), and the unexpected complex ones shown in Fig. 2(d)-(f), are described by the $C_j$ point symmetry group, where $j$ is the number of peaks. There are $j$ nearly degenerate solutions. For even $j$ there is a real symmetric anti-symmetric pair and $(j - 2)/2$ degenerate complex pairs. For odd $j$ there is a real symmetric solution and $(j - 1)/2$ degenerate complex pairs. Recall that only the real anti-symmetric solutions were expected.

Though in the attractive case the NLSE leads to collapse in three dimensions, it does not do so in one dimension, as may be shown by a variational argument. The expected anti-symmetric solution shown in Fig. 1(b) is lower in energy than its unexpected symmetric partner shown in Fig. 2(b), in contrast to intuition based on linear quantum mechanics. The constant amplitude, plane-wave solution is subject to modulational instability, and can collapse to any of a number of many-peaked nodeless solutions. This may help explain the disappearance of the condensate seen in the $^7\text{Li BEC}$ experiments.

In both cases, as the symmetry-breaking solutions may be placed anywhere on the ring, they have a high degeneracy, as is the case for symmetry-breaking vortex solutions in two dimensions. This leads to an entropy $S \sim k_B \log(L/(j\nu))$ where $L$ is the ring circumference, $j$ is the number of density peaks or notches, and $\nu$ is the size of a single soliton, about two to four times the healing length of the condensate. This is in direct analogy with the Kosterlitz-Thouless entropy.

In conclusion, we have discussed the physical form and properties of all stationary solutions to the 1D NLSE under periodic and box boundary conditions. In an adjoining article these results are extended into the experimentally realizable regime of pseudo-one-dimension.

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References