STO6 LECTURE 18

Note Title 4/19/2006

TODAY

Gaussian Channel (contd.)

- Capacity + Converse Coding Theorem
- Parallel Channels
- Colored Channel
- Feedback (?)

Recall from last lecture

Gaussian channel: $X \longrightarrow Y$

$\text{Var}[X] = P$

$$Y = X + Z \qquad Z \sim N(0, \sigma^2)$$

ind.

Capacity : $\max_{\substack{P_x \\ Var(x)=P}} \{ I(X;Y) \}$

$$= \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right)$$

Encoding :

- Pick $E : \{1 \dots 2^{nR}\} \rightarrow \mathbb{R}^n$ "at random"

$$(E(m))_i \sim N(0, P-\epsilon) \quad \text{i.i.d } \forall i, m.$$

- Decode : Given $\underline{Y} \in \mathbb{R}^n$ output

m if $\exists !\, m$ s.t.

$$\| E(m) - Y \|_2^2 \leq n(\sigma^2+\epsilon)$$

- $\Pr[\text{Decoding error}] \leq \Pr[E_0] + \Pr[E_1] + \Pr[E_2]$

- $E_0 \triangleq \|E(m)\|_2^2 \geq n \cdot P$; $\Pr[E_0] \to 0$
(LLN)

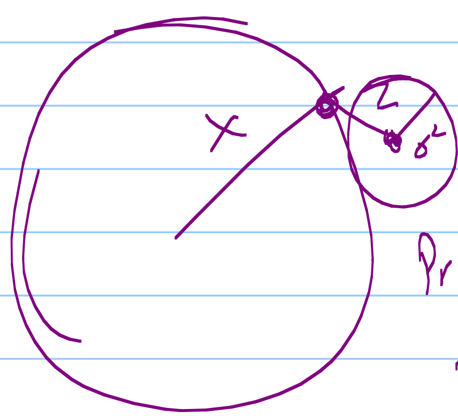
- $E_1 \triangleq \|E(m) - Y\|_2^2 \geq n(\sigma^2 + \epsilon)$; $\Pr[E_1] \to 0$
(LLN)

- $E_2 = \exists\, m'$ s.t. $\underbrace{\|E(m') - Y\|_2^2 \leq n(\sigma^2+\epsilon)}_{E_2(m')\,?}$.

- $\boxed{\Pr[E_2(m')] = ?}$

(Last time claim: $\Pr[E_2(m')] = 2^{-I(x;y)\cdot n}$)

Intuition



$E[\|Y\|] = n^2(\sigma^2+P)$

$\Pr[\text{fall in ball}] \approx \dfrac{\mathrm{Vol}(\sigma^2)}{\mathrm{Vol}(\sigma^2+P)}$

Formally Pick $\underline{X}, \underline{Y}$ from joint dist.
$\tilde{X}, \tilde{Y}$ iid from joint

$\Pr[\tilde{\underline{X}}, \underline{Y}]$ from joint typ. set =

$$\int_{\text{jointly typical set}} P_{\underline{X}}(\underline{x})\, P_{\underline{Y}}(\underline{y})\, d\underline{x}\, d\underline{y}$$

$$\leq \text{Vol(jointly typical set)} \cdot \max_{\underline{x} \in \text{typical set}} \{P_{\underline{X}}(\bar{x})\} \cdot \max_{\underline{y} \in \text{typical}} \{P_{\underline{Y}}(\bar{y})\}$$

$$\approx 2^{h(X,Y)\, n} \cdot 2^{-h(X)\cdot n} \cdot 2^{-h(Y)\cdot n}$$

$$\approx 2^{-I(X;Y)\cdot n}$$

$$\Pr[E_2] \leq 2^{Rn} \cdot 2^{-I(X;Y)\cdot n} \longrightarrow 0$$

if $R < I(X;Y)$

Adding them all have

- Capacity can be achieved.
- Method: Random "sphere packing."

Is Capacity an upper bound?

- Suppose $E: \{1 .. 2^{nR}\} \to \mathbb{R}^n$

with $\|E(m)\| \le n \cdot P \quad \forall m.$

- Then

$$nR = H(M) = I(M; Y^n) + H(M \mid Y^n)$$

$$= \underbrace{I(m; Y^n)}_{} + o(n) \quad \swarrow \text{Fano's Ineq.}$$

$$\Rightarrow \; \le I(X^n; Y^n) + o(n)$$

$$\leq \sum_{i=1}^{n} I(X_i ; Y_i) + o(n)$$

Gaussianity i.i.d s.t. P_i = power

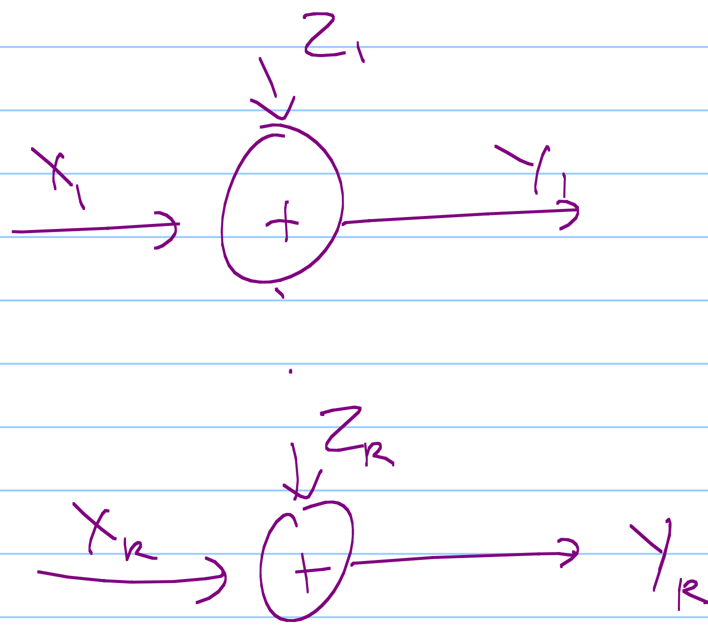
$$\leq \sum \frac{1}{2} \log \left(1 + \frac{P_i}{\sigma^2}\right) + o(n)$$

Jensen

$\sum P_i = nP$

$$\leq \frac{n}{2} \sum \log \left(1 + \frac{P}{\sigma^2}\right) + o(n)$$

Water-Filling (k parallel Gaussian Channels)



$Z_1 \sim N(0, \sigma_1^2)$

$\vdots$

$Z_k \sim N(0, \sigma_k^2)$

Constraint

$$\boxed{E\left[\sum X_i^2\right] \leq P}$$

How should we distribute power?

$$C = \max_{\substack{(X_1 \ldots X_k) \mid \\ E[\sum X_i^2] \leq P}} \left\{ I(Y_1 \ldots Y_k ; X_1 \ldots X_k) \right\}$$

Might as well make X_i's independent

$X_i \sim N(0, P_i)$ yields

$$C = \max_{\substack{P_1 \ldots P_k \\ \text{s.t. } \sum P_i = P}} \sum \frac{1}{2} \log\left(1 + \frac{P_i}{\sigma_i^2}\right)$$

Some Intuition

- $\sigma_1 = \sigma_2 \cdots \sigma_5 = 1$

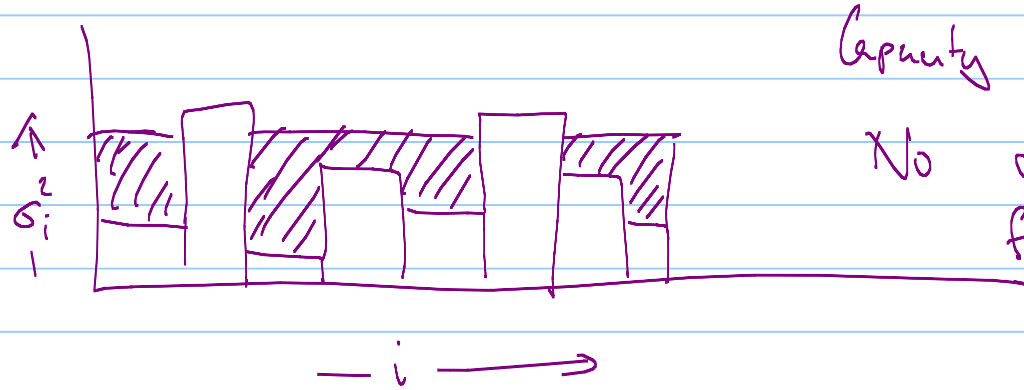
  $\sigma_6 \cdots \sigma_n = \infty$

  Should set $P_1 \cdots P_5 = \frac{1}{5}$.

- $\sigma_i \geq \sigma_j \Rightarrow P_i \leq P_j$

- Turns out optimal soln. is of the form

  $P_i = \max\{0, V - \sigma_i^2\}$ for some V



Capacity = ?

No simple form . . . .

Colored Noise

$Z_1 \ldots Z_k$ not independent but k-variate Gaussian.

Example:

$\tilde{Z}_1, \tilde{Z}_2$ i.i.d.

Actual Noise $Z_1 = \tilde{Z}_1$; $Z_2 = \tilde{Z}_1 + \tilde{Z}_2$

Used to model some limited memory.

What should we do?

Let K_Z = covariance matrix of Z

($k \times k$ matrix)

det K_x = covariance matrix of X

Cov. of Y = $K_x + K_z$

maximize $h(Y)$ s.t. $tr(K_x) \le P$

maximized by maximizing

$$|\det(K_x + K_z)|$$

$$Q Q^T = I$$

$$K_z = Q \Lambda Q^T$$

Diagonal

$$K_x = Q A Q^T$$

$$tr(A) \le P$$

then $\det(K_x + K_z)$ Reduces to

$$= \det(\Lambda + A) = ?$$

$\boxed{\text{Water filling}}$