| 6.441 Transmission of Information | February 23, 2006 |
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| Lecture 5 | |
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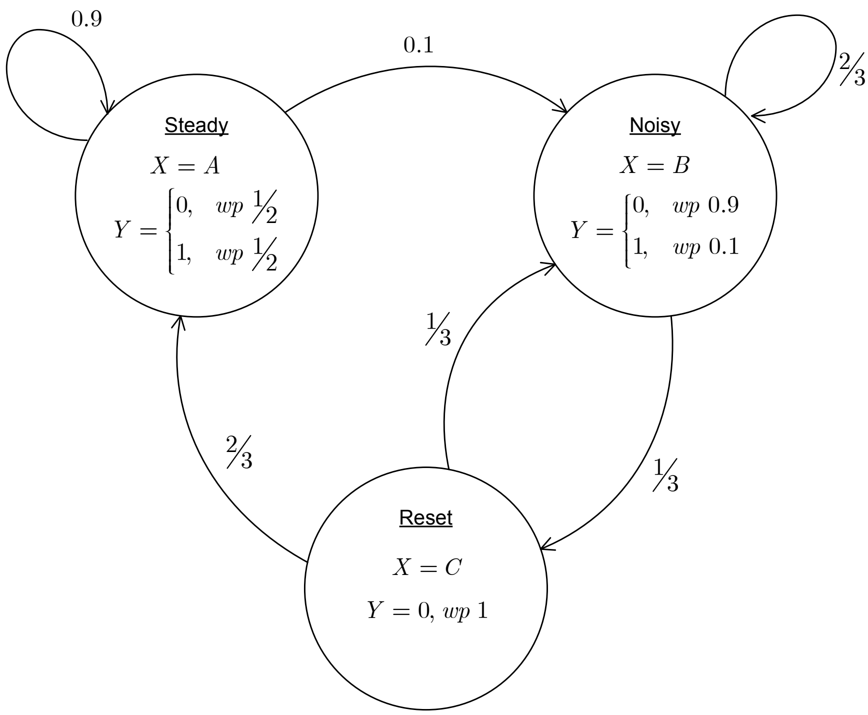
# 1 Introduction

## 1.1 Today's Topic

- Markov chains/processes
- Entropy rate of Markov chain

## 1.2 Motivating Example

**Example 1:** Let us start by considering the following example. What are the rates of $X$ and $Y$?




# 2 Stochastic Process

A stochastic process can be viewed as an infinite sequence of random variables, e.g., $X_{-n}$, $X_{-n+1}$, $\cdots$, $X_0$, $X_1$, $X_2$, $\cdots$, $X_n$, $\cdots$, whose distribution may be expressed by

$$\Pr[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n] \sim p(x_1, \cdots, x_n).$$

There are some meaningful and restricted classes of stochastic process.

**Definition 1 (Stationary Process)** *$\langle X_n \rangle_n$ is a stationary process if*

$$\Pr[X_1 = x_1, \cdots, X_n = x_n] = \Pr[\underbrace{X_{1+l} = x_1, \cdots, X_{n+l} = x_n}_{\text{time shift by } l}], \ \forall n, l, x_1, \cdots, x_n.$$

**Definition 2 (Markov Process/Markov Chain)** *$\langle X_n \rangle_n$ is a Markov chain if*

$$\Pr[X_n = x_n | X_1 = x_1, \cdots, X_{n-1} = x_{n-1}] = \Pr[X_n = x_n | X_{n-1} = x_{n-1}], \ \forall n, x_1, \cdots, x_n.$$

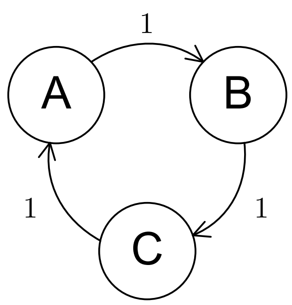
If $X_i \in \Omega$ and $\Omega$ is finite, then $\Pr[X_n = x_n | X_{n-1} = x_{n-1}]$ is just $|\Omega|^2$ entries for every $n$. But, can we describe it in finite terms? *No.*

**Definition 3 (Time Invariant Markov Chain)** *Markov Chain is time-invariant if*

$$\Pr[X_n = a | X_{n-1} = b] = \Pr[X_{n+l} = a | X_{n+l-1} = b], \ \forall n, l, a, b \in \Omega.$$

Time invariant Markov chain can be specified by distribution on $X_0$ and probability transition matrix $\boldsymbol{P} = [P_{ij}]$, where $P_{ij} = \Pr[X_2 = j | X_1 = i]$. Throughout the rest of lecture, time invariant Markov chain will be referred to simply as Markov chain (MC).

**Example 2:** Consider the following three-state MC. In this case, $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$




With $X_0 = A$, the resulting sequence will be "$ABCABCABC \cdots$." Note that this is *not* stationary because $\Pr[X_0 = A, X_1 = B, X_2 = C] = 1$ but $\Pr[X_1 = A, X_2 = B, X_3 = C] = 0$. Instead, $\Pr[X_1 = B, X_2 = C, X_3 = A] = 1$

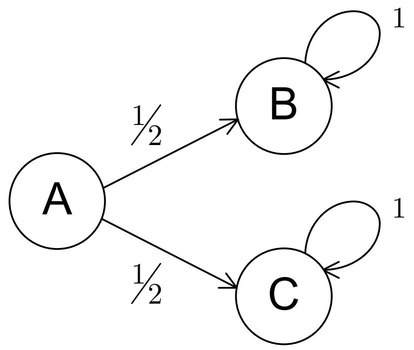
**Fact 1** *For every MC, $\exists$stationary distribution $\boldsymbol{\mu}$ on $X_0$ such that $\boldsymbol{\mu}$ and $\boldsymbol{P}$ define a stationary process. In the example 2, $\boldsymbol{\mu} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.*

Because

$$\begin{aligned} &\Pr[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n] \\ &\qquad = \Pr[X_1 = x_1] \cdot \Pr[X_2 = x_2 | X_1 = x_1] \cdots \Pr[X_n = x_n | X_{n-1} = x_{n-1}] \\ &\qquad = \Pr[X_1 = x_1] \cdot P_{x_1 x_2} \cdots P_{x_{n-1} x_n}, \end{aligned}$$

the overall distribution depends only on the distribution on $X_1$, which implies that the distribution $\boldsymbol{\mu}$ on $X_0$ is stationary if $\Pr[X_1 = i] = \mu_i (= \Pr[X_0 = i])$.

**Example 3:** Let us consider the following example:




In this case, $\mu_A = \mu_C = 0, \mu_B = 1$ is stationary, but $\mu_A = \mu_B = 0, \mu_C = 1$ is also stationary. More than one stationary distribution can be problematic, and this situation happens because the MC is reducible.

**Definition 4 (Reducibility of Markov Chain)** *1. Markov chain given by probability transition matrix $\boldsymbol{P}$ is reducible if $\boldsymbol{P}$ can be written as*

$$\left[\begin{array}{c|c} \boldsymbol{P_0} & \boldsymbol{P_1} \\ \hline \boldsymbol{0} & \boldsymbol{P_2} \end{array}\right],$$

*where $\boldsymbol{P_0}, \boldsymbol{P_2}$ are square matrices.*

*2. MC is irreducible if it is not reducible.*

In terms of graph structure, the "irreducible" and "aperiodic" characteristics can be interpreted as

- irreducible - strongly connected, ∃path from each state $i$ to state $j$.
- aperiodic - greatest common divisor of cycle lengths is 1.

**Theorem 2 (Perron-Frobenius's Theorem)** *Every (aperiodic) irreducible Markov chain has a unique stationary distribution.*

For stationary distribution, the probability distribution on $X_1$ should be the same as $\boldsymbol{\mu}$, the probability distribution of $X_0$. $\Rightarrow \Pr[X_1 = j] = \sum_{i=1}^{N} \mu_i P_{ij} = \mu_i$, where $N = |\Omega|$ and $\Omega = \{1, 2, \cdots, N\}$. If we use vector-matrix notation,

$$[\ \boldsymbol{\mu}\ ] \left[ \quad \boldsymbol{P} \quad \right] = [\ \boldsymbol{\mu}\ ], \tag{1}$$

and $\boldsymbol{\mu}$ corresponds to an eigenvector. For the example 1,

$$\boldsymbol{P} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 2/3 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}.$$

Theorem 2 implies that there exists a unique eigenvector with all entries non-negative. We can compute $\boldsymbol{\mu} = [\mu_1\ \mu_2\ \mu_3]$ using (1) and $\mu_1 + \mu_2 + \mu_3 = 1$. $\Rightarrow \boldsymbol{\mu} = [\frac{20}{32}\ \frac{9}{32}\ \frac{3}{32}]$.

## 3 Entropy Rate of Stochastic Process

There are two reasonable notions for measuring the uncertainty of $\mathscr{X} = \langle X_n \rangle_n$.

- Entropy rate:
$$H(\mathscr{X}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, \cdots, X_n) \text{ if the limit exists.}$$

- Entropy′ rate:
$$H'(\mathscr{X}) = \lim_{n\to\infty} H(X_n|X_1, \cdots, X_{n-1}) \text{ if the limit exists.}$$

**Theorem 3** *Entropy rate of a stationary stochastic process exists and equals entropy′ rate.*

$$H(\mathscr{X}) = H'(\mathscr{X}).$$

**Proof Idea** The following inequality can be used for the proof of the existence of $H'(\mathscr{X})$.

$$H(X_n|X_1, \cdots, X_{n-1}) \le H(X_n|X_2, \cdots, X_{n-1}) = H(X_{n-1}|X_1, \cdots, X_{n-1}).$$

For complete proof, refer to pp.64-65 of *Cover*. ■

**Theorem 4** *If irreducible MC has probability transition matrix* $\boldsymbol{P}$ *and stationary distribution* $\boldsymbol{\mu}$,

$$H(\mathscr{X}) = H'(\mathscr{X}) = -\sum_{i,j} \mu_i P_{ij} \log P_{ij}. \tag{2}$$

**Proof**

$$\begin{aligned} H'(\mathscr{X}) &= \lim_{n\to\infty} H(X_n|X_1, \cdots, X_{n-1}) \\ &= \lim_{n\to\infty} H(X_n|X_{n-1}) \\ &= H(X_2|X_1) \\ &= \sum_i \Pr[X_1 = i] \cdot H(X_2|X_1 = i) \\ &= -\sum_i \mu_i \sum_j P_{ij} \log P_{ij}. \end{aligned}$$

■

Using (2), $H(\mathscr{X})$ of the example 1 can be computed:

$$H(\mathscr{X}) = \frac{5}{8} H(0.9) + \frac{3}{8} H\left(\frac{2}{3}\right).$$

**AEP for Markov Chain:**

$$-\frac{1}{n} \log p(X_1, \cdots, X_n) \longrightarrow H(\mathscr{X}).$$

This doesn't follow from our law of large numbers because random variables may be dependent on each other.

**Hidden Markov Model:** Now, let us consider the rate of $\langle Y_n \rangle_n$ in the example 1. $H'(\mathscr{Y}) = \lim_{n\to\infty} H(Y_n|Y_1, \cdots, Y_{n-1})$, and is bounded by

$$H(Y_n|Y_1, \cdots, Y_{n-1}, X_1) \le H'(\mathscr{Y}) = \lim_{n\to\infty} H(Y_n|Y_1, \cdots, Y_{n-1}) \le H(Y_n|Y_1, \cdots, Y_{n-1}) \ \forall n.$$

(Try to prove the inequality at the left-hand side!) If we denote the interval between the upper and the lower bounds by $\epsilon_n$,

$$\epsilon_n = H(Y_n|Y_1, \cdots, Y_{n-1}) - H(Y_n|Y_1, \cdots, Y_{n-1}, X_1) = I(X_1; Y_n|Y_1, \cdots, Y_{n-1}),$$

and

$$\sum_{n=1}^{M} \epsilon_n = \sum_{n=1}^{M} I(X_1; Y_n|Y_1, \cdots, Y_{n-1}) \leq H(X_1).$$