LECTURE 19 4/23/2012

Note Title 4/22/2012

Today: More on Arithmetic Circuits

1. Finish [Baur-Strassen] [Partial Derivatives]

2. Circuits for computing determinant.
   [Berkowitz '84]
   (+ clarification on power of arithmetic circuits)

3. Depth reduction [Valiant, Skyum, Berkowitz, Rackoff]
   (as usual source = [Shpilka + Yehudayoff])

For notes on (1), see last lecture's prenotes.

2. Circuit for determinant.

- Clarification: Circuits are not universal for arithmetic computation. E.g. no circuit for "root-finding", or even "gcd"?

- So in principle it is possible to have a super-poly circuit lower bound for permanent + poly time algorithm.

- Circuit for determinant non-trivial.

- today [Berkowitz / Samuelson] circuit.

[Berkowitz]

Key Idea: Compute characteristic polynomial, which has inductive formula;

Characteristic poly:

- $P_M(\lambda) = \det(M - \lambda . I)$
- $\det(M) = P_M(0)$.

Inductive Formula

Notation: for poly $f(x) = \sum c_i x^i$

let $f^{[j]}(x) = c_d x^j + c_{d-1} x^{j-1} + \dots c_{d-j}$

Suppose

$$M = \begin{bmatrix} a & \bar{u} \\ \bar{v} & N \end{bmatrix}$$

then

$$P_m(\lambda) = a \cdot P_N(\lambda) - \bar{u} \cdot \left(\sum_{k=2}^{n} P_N^{[k-2]}(M) \cdot \lambda^{n-k} \right) \cdot \hat{v}$$

Proof omitted. Based on

- Cayley-Hamilton theorem $\quad P_m(m) = 0.$
- row column expansion ...

- Inductively compute $P_N(\lambda)$.
- Use coefficients + circuitry to compute $P_m(\lambda)$.

Depth Reduction

Theorem [VSBR]:

For any circuit Φ of size s, deg r

$\exists$ circuit Ψ of size poly(s,r)

depth polylog(s,r)

computing same polynomial.

Key Idea: • Use partial derivatives again.

• Compute $\{\partial_w(f_v)\}$ for all gates w, v in Φ

Notation: f_v = polynomial computed by v.

$\partial_w(f_v) = \begin{cases} \text{① Leave } w \text{ as formal variable.} \\ \text{② Take derivative wrt } w \\ \text{③ Evaluate at } f_w \text{.} \end{cases}$

- Stage i : • Compute all f_v of degree $\in \{2^i+1 \dots 2^{i+1}\}$
  - Compute all $\partial_w f_v$ for v, w s.t. $\deg(v) - \deg(w) \in \{2^i \dots 2^{i+1}\}$
    $\deg(v) \leq 2\deg(w)$.
  - Using one alternation of addition & multiplication of size $\mathrm{poly}(s, r)$.

<u>Key Lemma</u> : • Φ = homogenous circuit

- $G_m \triangleq \left\{ \begin{array}{l} \text{gates } t = t_1 \cdot t_2 \text{ with} \\ \deg(t_1), \deg(t_2) \leq m < \deg(t) \end{array} \right\}$

- $\forall\ v, w$ s.t. $\deg(w) \leq m < \deg(v) \leq 2\deg(w)$

$$f_v = \sum_{t \in G_m} f_t \cdot \partial_t f_v \ ; \quad \partial_w f_v = \sum_t \partial_w f_t \, \partial_t f_v$$

Proof of Lemma: Induction $\boxtimes$

[VSBR] theorem from Lemma:

Natural. Details Omitted.