A Crash Course on Coding Theory

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Topic: Bounds on Codes

This lecture will focus on limitations on the performance of codes. I.e., Upper bounds on rate/distance, or Lower bounds on block length.

Singleton bound

Thm: $n \geq k + d - 1$

- Note: Independent of q.
- Codes meeting the Singleton bound are called MDS codes (Max. Dist. Seperable). (Only) example: Reed-Solomon codes.

Proof (of Thm):

- Pick (any) $k-1$ coordinates and project code.
- Two codewords collide (PHP).
- Implies distance $\leq n - k + 1$.

Greismer bound

Thm: For linear codes, $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$.

In particular, $n \geq \frac{q}{q-1}d + k - \log_q d$.

Note: Strictly improves Singleton bound.

Proof: (for binary case)

$$\text{Let} \quad G = \begin{bmatrix} \overbrace{00\cdots0}^{n-d} & \overbrace{11\cdots1}^{d} \\ & \\ G' & G'' \\ & \end{bmatrix}$$

- Every row of G'' has $\geq \lceil \frac{d}{q} \rceil$ zeroes.
- G' generates $[n-d, k-1, \lceil \frac{d}{q} \rceil]_q$ code.
- Theorem follows.

Recall Hamming Balls

- $V(n,r,q) =$ "volume" of $B(\cdot, r)$ in Σ^n.
- Let $H_q(p)$ be q-ary entropy function.

$$H_q(p) = p \log_q \left(\frac{q-1}{p}\right) + (1-p) \log_q \left(\frac{1}{1-p}\right)$$

- Fact:

$$V(n, pn, q) \approx q^{H_q(p)n}$$

Packing (Hamming) Bound

Thm: $k \le \left(1 - H_q\left(\frac{1}{2} \cdot \frac{d}{n}\right)\right) n.$

Proof: Consider balls of radius $\frac{d-1}{2}$ around codewords.

- Balls don't intersect.
- Thus: $V(n, d/2, q) q^k \le q^n$
- Using approximation, get theorem.

Note: Codes meeting the inequality in proof tightly are called Perfect codes. e.g. Hamming codes (and only few others).

Compare with random linear codes:
(Letting $\delta = d/n$ and $R = k/n$)

$$1 - H_q(\delta) \quad \le \quad R \quad \le \quad 1 - H_q(\frac{\delta}{2}).$$

Intermission

- Have met Singleton, Griesmer and Hamming.
- Will soon meet Plotkin, Elias-Bassalygo, and Johnson.
- Will view MacWilliams and LP from afar.
- Why?

Comparing Bounds

- Obviously want the best bound for a given choice of parameters.
- Say fixed q, $R = k/n$, what is the best distance $\delta = d/n$?
- But relationship is not yet known!
- Further known relationships involve complicated functions - even if one is better, can verify this only by calculations?

Broad Issues

- Behavior at high rate? Hamming bound is good enough.
- Behavior at low-rate? Codes can't have $\delta > 1 - 1/q$, but Hamming bound can't prove this! Griesmer bound does, but only good for linear codes. Plotkin bound will work.
- Asymptotic behavior? Given k, ϵ, How does n behave is we want $\delta = 1 - 1/q - \epsilon$. Elias-Bassalygo bound will give a decent bound: $n = \Omega(k/\epsilon)$. LP bound gives the correct result $n = \Omega(k/\epsilon^2)$.

Bounds - Round II

Plotkin Bound:
If $d \geq (1+\epsilon) \cdot (1 - \frac{1}{q}) \cdot n$ then
# codewords $\leq 1 + \frac{1}{\epsilon}$.

Elias-Bassalygo Bound:
$R \leq 1 - H_q \left((1 - \frac{1}{q}) \cdot (1 - \sqrt{1 - \frac{q}{q-1}\delta}) \right)$.

Johnson Bound: If $\mathcal{C}$ is an $(n, ?, d)_q$ code then any Hamming ball of radius at most e contains at most nq codewords, provided

$$e/n < (1 - \frac{1}{q}) \cdot \left(1 - \sqrt{1 - \frac{q}{q-1}\delta} \right).$$

(Never mind the actual numbers for now.)

Proof Idea

- Will omit proof of Plotkin bound.
- Will start with Elias-Bassalygo and this will motivate the Johnson bound.
- Johnson bound: Proven via a geometric argument. (Proof + improved bound from [Guruswami+S.'01].)

Elias-Bassalygo Bound

- Pushes the packing bound.
- Go to larger radius.
- Suppose: Can prove that at most 4 balls of radius $e = 2d/3$ contain any one given point.
- Prveious argument gives:

$$V(n, 2d/3, q) q^k \leq 4 q^n.$$

- Lose almost nothing on RHS.
- Improve LHS (significantly).

Motivates the Johnson question.

Johnson Bound

Question: Given $\vec{r} \in \Sigma^n$, $(n,k,d)_q$ code $\mathcal{C}$.
How many codewords in $B(\vec{r}, e)$?

Motivation: (for binary alphabet)
How to pick a bad configuration?
I.e. many codewords in small ball.
W.l.o.g. set $\vec{r} = \vec{0}$.
Pick c_i's at random from $B(\vec{0}, e)$.

Expected' dist. between codewords = ?
Let $\epsilon = e/n$.
Codewords simultaneously non-zero on
ϵ^2 fraction of coordinates;
Thus distance $\approx (2\epsilon - 2\epsilon^2)n$.

Johnson bound shows you can't do better!

Hamming to Euclid

- Map $\Sigma \to \mathcal{R}^q$: ith element $\mapsto 0^{i-1}\ 1\ 0^{q-i}$.
- Induces natural map $\Sigma^n \to \mathcal{R}^{qn}$:
  - Maps vectors into Euclidean space.
  - Hamming distance large implies Euclidean distance large.

Argue: Can't have many large vectors with pairwise small inner products.

Hamming to Euclid (contd).

In our case:

Given: $c_1, \ldots, c_m$ codewords in Σ^n and $\vec{r} \in \Sigma^n$, s.t.

- $\Delta(c_i, \vec{r}) \le e$
- $\Delta(c_i, c_j) \ge d$

Want: Upper bound on m.

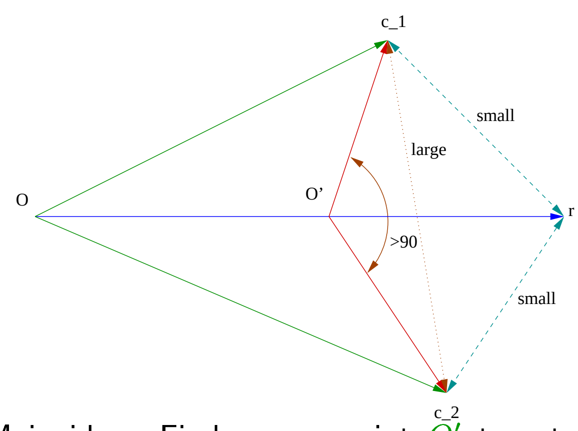
After mapping to $\mathcal{R}^{nq}$
(and abusing notation)

Given: $c_1, \ldots, c_m$ $\mathcal{R}^{nq}$ and $\vec{r} \in \mathcal{R}^{nq}$, s.t.

- $\langle \vec{r}, \vec{r} \rangle = n$.
- $\langle c_i, c_i \rangle = n$.
- $\langle c_i, \vec{r} \rangle \ge n - e$
- $\langle () c_i, c_j \rangle \le n - d$

Want: Upper bound on m.

Hamming to Euclid (contd).



Main idea: Find a new point O' to set as origin, such that the angle subtended by C_i and C_j at O' is at least $90°$.

Conclude: # vectors $\le$ dimension $= nq$.

Johnson bound (contd).

How to pick the new origin?

Idea 1: Try some point of the form $\alpha\vec{r}$.

Then $\langle c_i - \alpha\vec{r}, c_j - \alpha\vec{r}\rangle$
$= \langle c_i, c_j\rangle - \alpha\langle c_i\vec{r}\rangle$
$-\alpha\langle c_j, \vec{r}\rangle + \alpha^2\langle\vec{r}, \vec{r}\rangle$
$\leq (1-\alpha)^2 n + 2\alpha e - d$

Setting $\alpha = 1$, says: Need $e \leq d/2$.

Setting $\alpha = 1 - e/n$ yields:
Need $e/n \leq 1 - \sqrt{1-\delta}$.

(Not quite what was promised.)

Johnson bound (contd).

A better choice for origin.

Idea 2: Try some point of the form
$\alpha\vec{r} + (1-\alpha)\vec{Q}$,
where $\vec{Q} = (\frac{1}{q})^{qn}$.

Appropriate setting of $\alpha = 1 - e/n$ yields, the desired bound.

Back to Elias Bound

Plugging Johnson bound into earlier argument:

$$k \leq (1 - H_q(\epsilon))n + o(n),$$

where ϵ such that the Johnson bound holds for $e = \epsilon n$.

Importance:

- Proves e.g. No codes of exponential growth with distance $(1 - 1/q)n$.
- Decently comparable with existential lower bound on rate from random code.

MacWilliams Identities

Defn: Weight distribution of code is $\langle A_0, \ldots, A_n\rangle$, where A_i is # codewords of weight i.

- MacWilliams Identity determines weight distribution of code from weight distribution of its dual.
- Quite magical.
- Many nice consequences.

MacWilliams Identities

Thm:

- Let $A_0, \ldots, A_n$ wt. dist. of $\mathcal{C}$.
- Let $A'_0, \ldots, A'_n$ wt. dist. of $\mathcal{C}^\perp$.
- Let $W(y) = \sum_i A_i y^i$.
- Let $W'(y) = \sum_i A'_i y^i$.
- Then $W'(y) = \frac{(1+(q-1)y)^n}{|\mathcal{C}|} W\left(\frac{1-y}{1+(q-1)y}\right)$.

- Implications: Equating coefficients of y^i, get $n+1$ linear equations in $2(n+1)$ variables.
- Natural use, gives weight distribution of primal given dual or vice-versa.
- Interesting use: Can compute weight distribution of MDS codes!

MacWilliams Identities: Proof

(Will only do the Binary case)

Defn: The verbose generating function

(a) The generating function of a bit:
$W_b(x, y) = (1-b)x + by$

(b) The generating function of a word:
$W_c(x_1, y_1, \ldots, x_n, y_n) = \prod_{i=1}^b W_{c_i}(x_i, y_i)$

(c) The generating function of a code:
$W_{\mathcal{C}}(x_1, y_1, \ldots, x_n, y_n)$
$= \sum_{c \in \mathcal{C}} W_c(x_1, y_1, \ldots, x_n, y_n)$

E.g. if $\mathcal{C} = \{000, 011, 101, 111\}$, then
$W_{\mathcal{C}}(x_1, y_1, x_2, y_2, x_3, y_3)$
$= x_1x_2x_3 + x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$

MacWilliams Identities (contd).

Trivial Claim: Given $W_{\mathcal{C}}$, can compute $W_{\mathcal{C}^\perp}$.

Explicit version: (non-trivial)
$W_{\mathcal{C}}(x_1+y_1, x_1-y_1, \ldots, x_n+y_n, x_n-y_n)$
$= |\mathcal{C}| \cdot W_{\mathcal{C}^\perp}(x_1, y_1, \ldots, x_n, y_n)$

Proof steps:

Bit case:
$W_{b'}(x+y, x-y) = \sum_{b \in \{0,1\}} (-1)^{\langle b, b' \rangle} W_b(x, y)$.

Vector case:
$W_c(x_1+y_1, x_1-y_1, \ldots, x_n+y_n, x_n-y_n)$
$= \sum_{b \in \{0,1\}^n} (-1)^{\langle b, c \rangle} W_b(x_1, y_1, \ldots, x_n, y_n)$.

Proof (contd).

Code case:

$$W_{\mathcal{C}}(x_1+y_1, x_1-y_1, \ldots, x_n+y_n, x_n-y_n)$$

$$= \sum_{c \in \mathcal{C}} \sum_{b \in \{0,1\}^n} (-1)^{\langle b, c \rangle} W_b(x_1, y_1, \ldots, x_n, y_n)$$

$$= \sum_{b \in \{0,1\}^n} W_b(x_1, y_1, \ldots, x_n, y_n) \sum_{c \in \mathcal{C}} (-1)^{\langle b, c \rangle}$$

$$= |\mathcal{C}| \cdot W_{\mathcal{C}^\perp}(x_1, y_1, \ldots, x_n, y_n)$$

MacWilliams Identity follows using:

$$(1+y)^n W\left(\frac{1-y}{1+y}\right) = W_{\mathcal{C}}(1+y, 1-y, \ldots, 1+y, 1-y)$$

and $W'(y) = W_{\mathcal{C}^\perp}(1, y, \ldots, 1, y)$

MDS Codes

Fact: Dual of MDS code is MDS.

Proof: Along lines of Singleton bound.

Fact: MDS code of dim k has $(q-1)\binom{n}{k}$ codewords of minimum weight.

Proof: By inspection.

Consequence: Have values for $n+1$ variables out of $2(n+1)$ used in M.I. System turns out to have full rank.

Thm: # poly of degree $< k$ with w non-zero evaluations at n points is:

$$\binom{n}{w}\sum_{j=0}^{w+k-n}(-1)^j\binom{w}{j}(q^{w+k-n-j}-1)$$

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LP bound

- One more bound in literature.
- Strongest known bound.
- Analysis hard.
- So hard, one only has upper bounds on the LP bound.
- Current upper bound on LP bound is still far from random code or AG-code (so may not be optimal either).
- Will see LP later.
- However (only) bound proving that if $d = (\frac{1}{2}-\epsilon)n$, then $n = O(k/\epsilon^2)$. (Matches random code for small ϵ.)

LP bound

- Let $A_0, \ldots, A_n$ be dist. of $[n, ?, d]_q$ code.
- # codewords $= A_0 + \cdots + A_n$.
- Know $A_0 = 1$, $A_1 = \cdots = A_{d-1} = 0$.
- Further $A'_0 = 1, A'_1, \ldots, A'_n \geq 0$.
- How large can $A_0 + \cdots + A_n$ be under above conditions?
- Above is a linear program ... Gives best known bound [MRRW].
- Note: Extends to non-linear codes also.
  Define $A_i = \mathbb{E}_{c \in \mathcal{C}}[|S(c,i) \cap \mathcal{C}|]$,
  $S(c,i) =$ sphere of radius i around c.

Alon's proof for ϵ-biased spaces

Thm: Suppose have binary code with K codewords of length n s.t. no two are have distance less than $(\frac{1}{2}-\epsilon)n$ or greater than $(\frac{1}{2}+\epsilon)n$: Then $K \leq 2n$, provided $\epsilon \leq \frac{1}{2\sqrt{n}}$.

Proof:

- Map 0 to 1 and 1 to -1, and normalize so that vectors have unit norm.
- Then inner products lie between -2ϵ and 2ϵ.
- Let M be $K \times K$ matrix of inner products.
- M close to identity matrix and hence has rank close to that of identity matrix. Specifically: rank $\geq \frac{K}{1+4(K-1)\epsilon^2}$.
- On the other hand, rank$(M) \leq n$.