AM 106: Applied Algebra **Prof. Madhu Sudan**

Problem Set 0 - Solution

Assigned: Wed. Aug. 30, 2017 Due: Fri. Sept. 1, 2017 (11:59 pm)

- This problem set is optional and will not count for your grade. However, if you have not taken a prior proof-based math course, it is strongly encouraged that you complete and turn in the problem set for practice and feedback on doing proofs.
- You should submit your solutions via assignment page on the canvas website of the course.

Problem 1. (Proof by Contradiction) Joe the painter has 2016 cans of paint. Show that at least one of the following statements is true about Joe's paint collection.

- Among the cans, there are at least 32 of them with the same color.
- Among the cans, there are at least 66 different colors of paint.

Solution Assume for contradiction that *neither* statement is true. So we have that it is *not* the case that there is a color with at least 32 cans *and* it is *not* the case that there at least 66 different colors. In other words there are at most 65 different colors, and at most 31 cans of any one color. This gives a bound on the total number of cans of $31 * 65 = 2015$ which contradicts the assertion that there are 2016 cans. Hence our assumption must be false, or in other words at least one of the statements must be true.

Common Errors An error that I noted on a few solutions was the people argued that if the first statement was false then the second was true, and then continued to argue that if the second was false then the first was true. But the continuation is logically equivalent to the first part, and so the second part of the answer is really redundant. Knowing such aspects of logic is crucial to doing good proofs - so please do take note.

A few of you didn't use the fact that the number of cans of paint and the number of colors must be integers. So if the number at not at least 66 then it must be at most 65 (not 66). This difference is crucial to the final conclusion.

Problem 2. (Set Equality) Which of the following is true? Prove your answers.

- For every three sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap C$.
- For every three sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution The first statement is False. Take $A = a, B = C = \{\}$. Then $A \cup (B \cap C) = a$ whereas $(A \cup B) \cap C = \{\}$ and so the two expressions are not equal.

The second statement is True. Consider $x \in A \cup (B \cap C)$. Such an x must either be from A or from $B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in (A \cup B) \cap (A \cup C)$. If

$x \in B \cap C$ then $x \in B$ and $x \in C$ and so $x \in A \cup B$ and $x \in A \cup C$. Thus in this case also we have $x \in (A \cup B) \cap (A \cup C)$. We conclude $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To conclude we need to show the other direction of the containment i.e., $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Consider $xin(A \cup B) \cap (A \cup C)$. We have $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$ then we also have $x \in A \cup (B \cap C)$. If $x \notin A$ then we must have $x \in B$ (since $x \in A \cup B$) and $x \in C$ (since $x \in A \cup C$). So $x \in B \cap C$ and so, in this case also we have, $x \in A \cup (B \cap C)$. We conclude $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ and so the two sets are equal.

Common Errors One common error was in the misreading of the question itself. Several students interpreted "Which of these statements is true" to mean "Which one of these statments is true". Note that there was no intention in the question to suggest that either at least one or at most one of the statements was true. So the task was to determine, for each of the statements, whether it was true or not and then to prove it.

For the first statement a common form of answer was "Here is a general condition which would imply that the statement is false." For example if $A \setminus C \neq \{\}$. While it is "obvious" that there are sets satisfying this condition, it is better to prove the "obvious" fact by giving example sets; and then it is trivial to prove the statement is false by working out what the left hand side and right hand side are. In general an explicit counterexample is better (as a proof) than a condition under which the statement is false, even when the latter might capture a better understanding of why the statement is false.

A few of you gave explanations via Venn diagrams. While these are good for your own reasoning, these are not good for written proofs. Venn diagrams can lead to errors and a proof as the above is the right way to write proofs.

Problem 3. (Induction) The Fibonacci numbers $F_0, F_1, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Thus the sequence (starting at F_0) is $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Prove by induction that for $n \geq 2$, $F_n \geq \varphi^{n-2}$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio.

Solution We prove the statement by (strong) induction on n. The base cases are with $n = 2$ and $n = 3$. We have $F_2 = 1 = \varphi^0$ as desired. We also have $F_3 = 2 > \varphi^1$. Now assume $n \geq 3$ and by the inductive hypothesis we have that $F_j \geq \varphi^{j-2}$ for every $j \in \{2, 3, \ldots, n\}$. We now prove the statement for $n + 1$. Note that we have $F_{n+1} = F_n + F_{n-1} \geq \varphi^{n-2} + \varphi^{n-3} = \varphi^{n-3}(\varphi + 1)$, where the inequality comes from the strong inductive hypothesis. Now we use the property of the golden ratio that $\phi + 1 = \phi^2$ and this lets us conclude that $F_{n+1} \geq \varphi^{n-3}(\varphi + 1) = \varphi^{n-3} \cdot \varphi^2 = \varphi^{n-1}$ as desired.

Common Errors A common error was not proving the "base case" with F_3. This is important since the inductive step uses the inductive hypothesis for both F_n and F_{n-1}.

In general an inductive proof should be thought of as a formal way of detecting a pattern. We prove the assumption for F_2 first, then F_3, then F_4 and so on. At some point we should detect a pattern in the proof and then try to use symbols (like n and $n - 1$) to formalize the pattern. In this case we start with F_2 and just prove the result from scratch. Then we move to F_3 and can try to use the recursive formula $F_3 = F_2 + F_1$, but we don't have any assertions about F_1. So we just prove the assertion from scratch. Then we look at F_4 and reason about it by using $F_4 = F_3 + F_2$ and try to use our inequalities about F_2 and F_3. This works and the pattern is also visible, allowing us to prove the statement for all higher n now.

Problem 4. (Incorrect Induction) What is the wrong with the following proof by induction?

Claim: In every set of n students, all students have the same height.
"Proof" by Induction:

- Base Case: For every set of size 1, the claim is clearly true (all the students in that set have the same height).
- Induction Step: Assume that the claim is true for sets of k students (this is the induction hypothesis), and we'll prove that it also holds for sets of $k+1$ students. Consider an arbitrary set S consisting of $k+1$ students, say $S = \{p_1, \ldots, p_{k+1}\}$. Let $S' = \{p_1, \ldots, p_k\}$. Since $|S'| = k$, our induction hypothesis tells us that all students in S' have the same height. So now we only need to show that p_{k+1} has the same height too. To do this, consider the set $S'' = \{p_2, \ldots, p_{k+1}\}$. Since $|S''| = k$, the induction hypothesis also tells us that all students in S'' have the same height. In particular, p_{k+1} has the same height as p_2, and hence the same height as all students in S'.

Solution The error in the proof above is when $k = 1$. In that case $S' = \{p_1\}$ and $S'' = \{p_2\}$. The inductive hypothesis applied to S' does not imply that the height of p_1 equals the height of p_2 and so we don't have $h(p_1) = h(p_2) = h(p_{k+1})$ in this case.

Common Errors Several people objected to the fact that we were applying induction to two different sets of size k at the same time (both S' and S'') and that this violated the principle of induction. But this on its own is not a violation. Others thought the while we had proved all people in S' have the same height and all people in S'' have the same height, we had not proved that these heights were the same. But we do argue about this explicitly in the "incorrect proof" by supposing that p_2 is in both sets. The only error is the one pointed to in the solution; and this is a valuable lesson - it only takes one error to prove completely absurd things.