AM 106: Applied Algebra **Prof. Madhu Sudan**

Problem Set 6

Assigned: Wed. Oct. 18, 2017 Due: Wed. Oct. 25, 2017 (8:00 AM)

- You may submit your solutions via assignment page on the canvas website of the course.
- For collaboration and late days policy, see course website at `http://madhu.seas.harvard.edu/courses/Fall2017`
- Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details. Justify your answers except when otherwise specified.

Problem 1. (Homomorphisms) Give three distinct homomorphisms from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{84}$. For each, identify the kernel and image. (Hint: how does $\varphi(x)$ relate to $\varphi(1)$? And what can we say about the order of $\varphi(1)$?)

Problem 2. (Normal Subgroups of Small Index)

1. Show that if H is a subgroup of G of index 2, then H is normal in G.
2. **(Optional, 0 points)** Show that if G has no subgroups of index 2, then every subgroup of index 3 is normal. (Hint: Multiplication on the left by $g \in G$ permutes the cosets of H. This gives rise to a homomorphism $\varphi : G \to S_3$. Reason about $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$, and $\varphi^{-1}(A_3)$.)

Problem 3. (Cautions with Normality and Factor Groups)

1. Give an example of a group G and normal subgroup N such that $G/N \times N \ncong G$.
2. Give an example of a group G and subgroups N and H such that $H \lhd N \lhd G$, but $H \ntriangleleft G$.

Justify your answers.

Problem 4. (Factor Groups and Homomorphisms) For each of the following groups G and subsets $H \subseteq G$, determine whether H is a normal subgroup of G. If yes, then find a familiar group G' such that $G/H \cong G'$. Prove that $G/H \cong G'$ by giving an appropriate homomorphism from G to G'.

1. $G = \mathbb{Z}$, $H = \{\text{prime integers}\}$.
2. $G = S_5 \times S_5$, $H = \{(\sigma, \sigma) : \sigma \in S_5\}$.
3. $G = \mathbb{C}^*$, $H = S^1 = \{z \in \mathbb{C}^* : |z| = 1\}$.