Problem 1. (Induction) Consider the following one-dimensional variant of “Game of Life”. The process starts at time 0 with one particle sitting at the origin $x = 0$. At each time step a particle at location $x = i$ splits into two particles with one placing itself at location $i + 1$ and the other at location $i - 1$. But if two particles attempt to place themselves at the same location, they annihilate and die. (See Figure below for an illustration.)

![Particle evolution for 3 steps.](image)

1. How many particles are there at time $t = 65535$? Prove your answer.
   (Hint: Play with the particles to form a hypothesis about how the evolution goes. This hypothesis might need to be strengthened to make it suitable for induction. State the hypothesis clearly, and then prove it by induction.)
   (Warning: This problem may be harder than the rest of this problem set - and if you are stuck it might be a good idea to do the other problems first and return to this at the end.)

2. (Extra credit, need not be turned in): Give a formula expressing the number of particles at time $t$ for general $t$. (No need to prove your answer.)
Problem 2. (Asymptotic Notation) True or False? Briefly justify your answers in one sentence each. Your answers should go back to the definitions of $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$. For example, the definition of $O(\cdot)$ says that $f(n) = O(g(n))$ if there exist $c$ and $n_0$ such that for every $n \geq n_0$ it is the case that $f(n) \leq c \cdot g(n)$. So, if the answer is true, give the values of $c$ and $n_0$ such that the statement holds; Or if the answer is false, explain why no $c$ or $n_0$ would work.

1. $5n + 6 = O(n)$.
2. $n^2 = O(n^3)$.
3. $n^2 = \Omega(n^3)$.
4. $n = O(\log^2 n)$.
5. $\ln n = \Theta(\log_2 n)$.
6. $5^n = 3^{O(n)}$.

Problem 3. (GCD)

1. Consider the sequence of fractions $(4n + 1)/(6n + 1)$ for $n = 1, 2, \ldots$. That is, the sequence $5/7, 9/13, 13/19, \ldots$. Prove that all of these fractions are written in lowest terms. (Hint: GCD is a linear combination.)
2. (Extra Credit) Prove the same for the sequence $(3n + 1)/(6n + 5)$.

Problem 4. (Equivalence Relations) Which of the following are equivalence relations? If it is an equivalence relation, describe the equivalence classes. If it is not, which properties fail?

- Domain: Finite subsets of $\mathbb{N}$. Relation: $A \sim B$ if $|A| = |B|$.
- Domain: Functions $h: \mathbb{N} \to \mathbb{N}$. Relation: $f \sim g$ if $f(n) = O(g(n))$.
- Domain: The set of positive integers. Relation: $a \sim b$ if $ab$ is a perfect square.
- Domain: $\mathbb{N} = \{0, 1, 2, \ldots\}$. Relation: $a \sim b$ if $ab$ is a perfect square. (We consider 0 to be a perfect square.)

Problem 5. (Modular Exponentiation) Follow the instructions at http://madhu.seas.harvard.edu/courses/Fall2019/sage-inst.html to create an account for yourself on SAGE and get comfortable using it. Use SAGE to solve this problem and submit a copy of your notebook showing all your work. You should submit an annotated pdf version on Canvas.

- What is the largest value of $n \in \mathbb{N}$ for which you can get SAGE to calculate $3^n$?
- Calculate $3^n \mod 10^6$ for each $n \in \{2^{10} = 1024, 2^{20} = 1048576, 2^{30} = 1073741824, 2^{100}, 2^{100} + 2^{30} + 2^{20} + 2^{10}\}$.
- Calculate $5^n \mod 11$ for each $n \in \{0, 1, \cdots, 25\}$ (look for a pattern!) and $n = 8^{2^{100}}$. 

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