CS 121: Lecture 15
More Uncomputability

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How to contact us
- The whole staff (faster response): CS 121 Piazza
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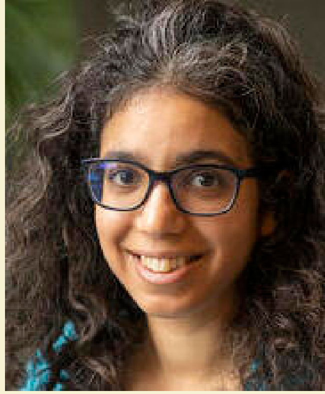
Announcements:



- Advanced Section: Nada Amin: Uncomputability & PL Design
- Thanks for feedback.
    - Confirm – are breakouts no good?
    - TFs scouring the feedback also!
- Sections: Week 7 cycle start, material on canvas (as usual).

Where we are:

Part I: Circuits:
Finite computation,
quantitative study

Part II: Automata:
Infinite restricted computation,
quantitative study

Part III: Turing Machines:
Infinite computation, qualitative study

Part IV: Efficient Computation:
Infinite computation, quantitative study

Part V: Randomized computation:
Extending studies to non-classical algorithms

Review of last lecture

- # of functions = uncountable
- # of computable functions = countable.
- So … ∃ an uncomputable function
- Further $\mathrm{Cantor}(M) = \overline{M(M)}$ uncomputable

This lecture (& next)

- Uncomputability much more pervasive
- “Intent of a program” uncomputable

Today: HALT is uncomputable

- Definition: HALT(M,x) = 1 if M halts on input x; 0 otherwise.

- 2 Proofs:
  - Diagonalization
  - Reduction from CANTOR

Proof 1 (Direct Diagonalization):

- Let A be a TM that solves HALT, i.e., $\forall M, x,\ \ A(M, x) = \mathrm{HALT}(M, x)$
- Consider the following Algorithm (which has equivalent TM – HOCAEIT)

$B(z)$:
Compute $A(z, z)$
If $A(z, z) = 1$ then loop forever
Else Halt and output 1.

- Note: We are defining B but not running it! It does not have to halt (in fact crucial that it does not on some inputs.
- Key point: B is a TM.

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- Consider B

$B(z)$:
Compute $A(z,z)$
If $A(z,z) = 1$ then loop forever
Else halt and output 1.

- What is $A(B,B)$?
  - Case 1: $A(B,B) = 1 \Rightarrow$ (by correctness of A) B halts on input B
    $\Rightarrow$ (by construction of B) B loops forever $\Rightarrow$ Contradiction.

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  - Case 1: $A(B, B) = 1 \Rightarrow$ (by correctness of A) B halts on input B
    $\Rightarrow$ (by construction of B) B loops forever $\Rightarrow$ Contradiction.
  - Case 2: $A(B, B) = 0 \Rightarrow$ (by correctness of A) B does not halt on input B
    $\Rightarrow$ (by construction of B) B halts on B (outputs 1) $\Rightarrow$ Contradiction!

Thoughts:

- Very slick!
- But just an implementation of Diagonalization. (Note $B(B)$; $A(z, z)$ ...)
- Food for thought: What happens if A does not always halt but correctly determines $\mathrm{HALT}(M, x)$ on inputs where it halts?

Proof 2: (General) Reduction

- Reductions: Key theme in Computer Science
  - Function F reduces to G ($F \leq G$) if algorithm for G implies algorithm for F
  - How to prove it?

```
Alg-F(x):
Blah Blah Blah
z = Alg-G(y)
Blah blah blah
```

    - Build algorithm for F using Alg-G as subroutine.
    - Alg-F correctly computes F if Alg-G correctly computes G
- Usual Interpretation: Positive:
  - Somebody builds tools for mean, median; I just invoke it on my data with wrapper.
- Our Use: Negative:
  - Start with F known not to have algorithm. Infer G does not!
    - Do you remember any so far in this course?

Example: $\mathbf{HALT}$ uncomputable

- Recall CANTOR uncomputable.
- Will use this to prove HALT uncomputable.
- So what do we need to do?

Alg-$F(x)$:
Blah Blah Blah
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- Recall CANTOR uncomputable.
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Alg-CANTOR(x):
Blah Blah Blah
$z =$ Alg-HALT(y)
Blah blah blah

ALG-CANTOR

- Recall $\mathrm{CANTOR}(M) = \overline{M(M)}$

```
Alg-CANTOR(M):
   b ← Alg-HALT(M, M)
   If b = 0 output 1
   Else run M on M and let output be c
        Output c̄
```

- Claim 1: Alg-CANTOR always halts if Alg-HALT correct.
- Claim 2: Alg-CANTOR correctly computes CANTOR.

- Claim 1+Claim 2: Alg-CANTOR computes (the uncomputable function) CANTOR if Alg-HALT exists $\Rightarrow$ Alg-HALT does not exist $\Leftrightarrow$ HALT uncomputable.

What did we prove?

- CANTOR $\leq$ HALT ? Or HALT $\leq$ CANTOR?

(Basic) Reduction

- For many problems we will use a very basic reduction (even simpler than $\mathrm{CANTOR} \leq \ \mathrm{HALT}$)

Alg-F(x):
 $y = R(x)$
 $Return$ Alg-G(y)

Example:

- $E(M) = 1 \Leftrightarrow \ \forall x, \quad M(x) = 0$ or M does not halt on x

- $\mathrm{HALT} \leq E$

Alg-HALT(M, x):
Define M_x as follows:

> $M_x(z)$: Ignore z,
> output 1 if M halts on x
> output 0 o.w.

Return Alg-E(M_x)

Section + Next Lecture

- More Uncomputability + Reductions
  - HALT-ON-ZERO
    - H-O-Z(M) = 1 if M accepts “” and 0 otherwise.
    - Moral: It is not the infinity of inputs that makes HALT hard!
  - Rice’s theorem
    - Every non-trivial semantic property of algorithms is uncomputable!