LECTURE 11

Note Title 2/27/2016

(Part 2): Proof of Lemma 3

Recall Lemma 3, $(X, Y, Z) \in \{0,1\}^3$

$I(XY; \Pi \mid Z) > 0$ if Π computes $X \wedge Y$ w.p. $1-\epsilon$ on every input

$Z \leftarrow \text{Unif}(\{0,1\})$

$\tilde{X}, \tilde{Y} \leftarrow \text{Unif}(\{0,1\})$

$X = \tilde{X} \wedge Z$; $Y = \tilde{Y} \wedge \bar{Z}$

Proof Intuition

- Let $\Pi^{00}, \Pi^{01}, \Pi^{10}, \Pi^{11}$ be r.v.'s corresponding to transcript on $XY = \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$

- Question 1: Why can't they be identically distributed? (if they were $I(XY; \Pi \mid Z) = 0$)

– Answer: Rectangle Property.

Challenge: Make this quantitative

Rectangle Property

(1) if $\pi((x, R_A), (y, R_B)) = t = \pi((x', R_A'), (y', R_B'))$

then

$$\pi((x, R_A), (y', R_B')) = t = \pi((x', R_A'), (x, R_B))$$

(2) $\Rightarrow$ $Pr[\pi^{xy} = t] \cdot Pr[\pi^{x'y'} = t]$

$$= Pr[\pi^{x'y} = t] \cdot Pr[\pi^{xy'} = t]$$

(3) Does above imply some distance property?

What measure of distance?

Correct measure "Hellinger Distance"

Hellinger Distance

$$h_2(P,Q) \triangleq \frac{1}{2}\, \ell_2(\sqrt{P}, \sqrt{Q})$$

$$= \frac{1}{2}\left(\sum_x \left(\sqrt{P(x)} - \sqrt{Q(x)}\right)^2\right)^{1/2}$$

$$= \frac{1}{2}\left(2 - 2\sum_x \sqrt{P(x)\cdot Q(x)}\right)^{1/2}$$

So if $P(x)\, Q(x) = P'(x)\, Q'(x) \quad \forall\, x$

then $h_2(P,Q) = h_2(P', Q')$

Apply to Π^{00}, Π^{11} & Π^{10}, Π^{01}, we get

Lemma: $h_2(\Pi^{00}, \Pi^{11}) = h_2(\Pi^{01}, \Pi^{10})$ — (1)

By correctness

$$\text{tvd}(\Pi^{00}, \Pi^{11}) \geq 1 - 2\epsilon \quad \text{— (2)}$$

---

(1) + (2) $\Rightarrow$ $I(XY ; \Pi^{XY} \mid Z) > 0$

(inf. theory drill).

Information Theory Drill

1. TVD high $\Rightarrow$ h_2^2 high
   $\Rightarrow$ $h_2^2(\Pi^{00}, \Pi^{11})$ high

2. $h_2^2(\Pi_{01}, \Pi_{10})$ high $\Rightarrow$
   $TVD(\Pi_{10}, \Pi_{01})$ high

Proof of 1 & 2 follows from

$$h^2(P,Q) \leq TVD(P,Q) \leq \sqrt{2}\, h(P,Q)$$

(3) $TVD(\pi_{01}, \pi_{10}) \geq \text{high}$

$$\left.\begin{cases} \Rightarrow TVD(\pi_{01}, \pi_{00}) \geq \frac{\text{high}}{2} \\ \Rightarrow I(0Y ; \pi^{0Y} \mid Z=0) \geq \text{high}' \end{cases}\right\} (1)$$

OR

$$\begin{cases} \Rightarrow TVD(\pi_{00}, \pi_{10}) \geq \frac{\text{high}}{2} \\ \Rightarrow I(X0 ; \pi^{X0} \mid Z=1) \geq \text{high}' \end{cases}$$

<u>Proof of (1)</u>

Let $\pi_{1/2} = \frac{1}{2}(\pi_{00} + \pi_{10})$

(1) $I(\cdot,\cdot)$ related to divergence

$$I(Y ; \pi^{0Y}) = \frac{1}{2}\left[D(\pi_{1/2} \,\|\, \pi_{01}) + D(\pi_{1/2} \,\|\, \pi_{00})\right]$$

(2) Divergence lower bounded by TVD (Pinsker)

- $D(\Pi_{1/2} \,\|\, \Pi_{01}) \geq \frac{1}{2} \mathrm{TVD}(\Pi_{1/2}, \Pi_{01})^2$

$$= \frac{1}{2}\left(\frac{\mathrm{TVD}(\Pi_{00}, \Pi_{01})}{2}\right)^2$$

- Similarly $D(\Pi_{1/2} \,\|\, \Pi_{00}) \geq \frac{1}{2}\left(\frac{\mathrm{TVD}(\Pi_{00}, \Pi_{01})}{2}\right)^2$

$\Rightarrow I(Y; \Pi^{0,1}) \geq \frac{1}{2}\left(\frac{\mathrm{TVD}(\Pi_{00}, \Pi_{01})}{2}\right)^2$ $\boxtimes$