LECTURE 13

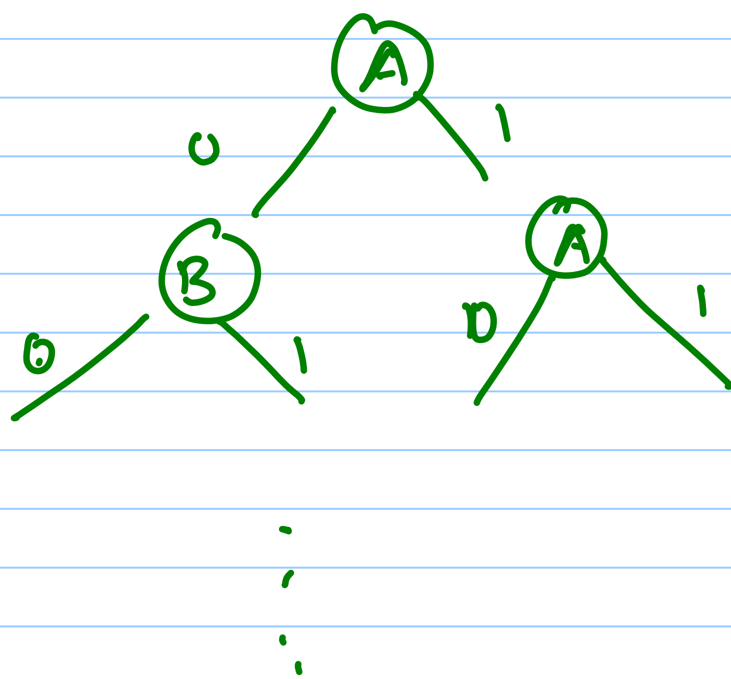
Note Title 3/8/2016

TODAY

- Compression of Low Information Protocols.
- Direct Sum

Protocol trees, Priors, Information Cost

- Wlog no public randomness in protocol;
  All uncertainties due to $(X, Y) \sim \mu$;
  $R_A, R_B \sim \text{Unif}$.
- Protocol = tree



- Alice's view:
  - knows which fork to take on A nodes, given R_A.

- knows prob. [fork] if R_A not fixed.

- has guess on Prob [fork]

Thus Protocol described by tree with three "probabilities" at each node

P_v : Prob of going right | v

P_v^A : Prob. Alice thinks we'll go right | v

P_v^B : Prob. Bob thinks we'll go right | v.

$$P_v \approx \Pi_i \mid XY\Pi_{<i}$$

$$P_v^A \approx \Pi_i \mid X\Pi_{<i}$$

$$P_v^B \approx \Pi_i \mid Y\Pi_{<i}$$

(first = one of second / third depending on who speaks)

Information Cost

$$= I(\Pi; X \mid Y) + I(\Pi; Y \mid X)$$

$$= \sum_i \underbrace{\left[H(\Pi_i \mid X \Pi_{<i}) + H(\Pi_i \mid Y \Pi_{<i}) - 2H(\Pi_i \mid X Y \Pi_{<i}) \right]}_{v_i}$$

Information Cost $\ll$ Communication Complexity

$\Rightarrow$ typical v_i small $\approx \frac{I}{C}$

- Keep this in mind...
- Extreme Case: $I = 0$
- Want zero comm. protocol ... How?

  Idea: Sample leaf of tree with public randomness.

Fact 0: if Alice & Bob know distribution D over Ω, can sample $\omega \sim D$ with zero communication.

Fact 1: if Alice & Bob have distribution P, Q over Ω, they can sample $\omega_A \sim P$, $\omega_B \sim Q$ with

$$\Pr[\omega_A \neq \omega_B] \leq 2 \cdot TVD(P, Q)$$

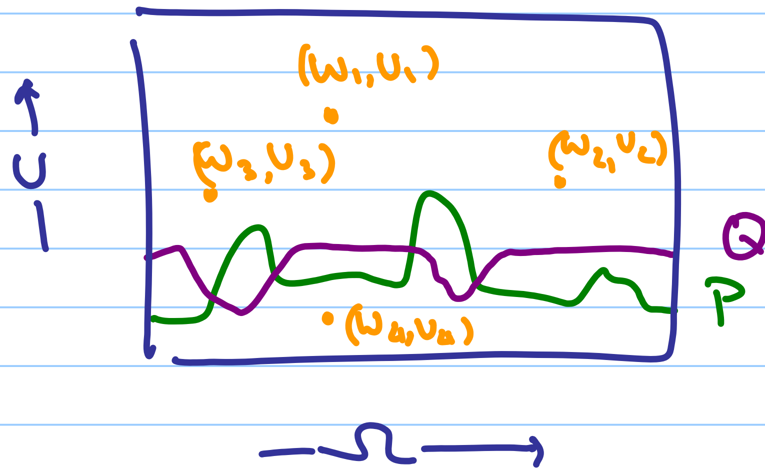
with zero communication.

Correlated Sampling

Protocol: $R = (\omega_i, u_i)_{i=1,2,\dots}$

$\omega_i \in \Omega$; $u_i \in [0,1]$

Pictorially



- $w_A = w_{i_A}$ where $i_A = \arg\min_i \{ v_i < P(w_i) \}$

- $w_B =$ similar.

- Analysis: Exercise.

Conclude: if $I = \epsilon$ then can simulate communication with O bits & small error

how much?

(surprisingly large amounts!)

Example: - Say Alice is the only one talking

- Bob thinks each fork is 50-50.

- Alice picks left w.p. $\frac{1}{2} - \delta$, & right w.p. $\frac{1}{2} + \delta$.

(or other way based on X_i)

- $IC = k \cdot \delta^2 = \epsilon$

depth of tree

- $\Pr[\text{agreement on bottom leaf}]$

$$\approx \delta \cdot k = \sqrt{\epsilon \cdot k}.$$

$\sqrt{I}$ $\quad$ $\sqrt{C}$ $\quad \boxed{!}$

✓ Right behavior.

✗ Wrong place (Error, not communication)

✗ Wrong type: Impossibility not achievability

Path Compression

High Error, Zero Comm. Protocol

$\Rightarrow$ Low error, Some Comm. Protocol.

Protocol:

for every node v,

Alice chooses successor according P_v^A

Bob chooses successor according P_v^B

ℓ_A = Alice's leaf ;

ℓ_B = Bob's leaf ;

u = least common ancestor (ℓ_A, ℓ_B)

Stop if u = leaf

Else repeat with u as root.

Claim 1 (Easy):

Comm. Cost per iteration = $O(\log C)$

(Equality testing + binary search)

Claim 2 (Harder):

$\#$ iterations $= O(\sqrt{IC})$.

Analysis

- Let Z_i = Indicator variable if there is disagreement at i^{th} level.

- $E[Z_i] = TVD(\Pi_i | X\Pi_{<i}, \Pi_i | XY\Pi_{<i})$

(if Bob speaks)

$$\leq 2\sqrt{D(\Pi_i | X\Pi_{<i} \| \Pi_i | XY\Pi_{<i})}$$

$$\leq 2\sqrt{v_i}$$ (as defined on page 5)

- Expected # iterations

$$= \sum_i E[Z_i]$$

$$\leq \sum_i 2\sqrt{v_i}$$

$$\leq \sqrt{C} \cdot O\left(\sqrt{\textstyle\sum_i v_i}\right) \qquad \text{[Cauchy Schwartz]}$$

$$= O\left(\sqrt{C \cdot T}\right).$$

⊗