Today:

- Space complexity (quick defn.)
- Non-Determinism
  - NP
  - NEXP?
  - NPSPACE
  - NL

- Complete Problems
  - NP vs. CoNP vs. P
  - NL vs. CoNL vs. L

1) Space Complexity of L.

Algorithm: Gets to (re)read input.

- Limited memory $S(n)$ bits
- Outputs 0/1 answer
  - Longer output?
  - Can write on output tape...
Exercise / Set 1:

if $f_1$ computable in space $S_1$,

$f_2$ in $S_2$

$\Rightarrow f_1 \circ f_2$ in space $S_1 + S_2 + O(1)$.

Non-determinism:

At any moment, the algorithm can make a choice.

Q: A sequence of choices that leads the algorithm to accept.

leads to Non-det. Any $\text{NSPACE}$
Alt / Equiv view of NP

- "Pair languages": Instance are pairs $\leq 30.13^* \times 30.13^*$

- $A \in \text{NP}$ if $\exists$ pair language $B \in \text{P} \leq \text{NP}$, no $x \in A \iff \exists y \text{ st. } (x, y) \in \Phi B$
  \[
n \leq |x| + n_0
\]

- $L \in \text{NEXP}$? What changes?

Alt / Equiv view of NL

- $A \in \text{NL}$ if $\exists$ log-time algorithm $B(x, y)$,
  - that accesses "y" only left to right,
  - uses $O(\log (|x|))$ space
  \[
  \exists x \in A \iff \exists y \text{ st. } B(x, y) = 1.
  \]
Aside: Quantifiers & Pair Languages

Very useful notation; good to start thinking in such terms.

Example

\[ A \in \text{P/poly} \text{ if } \exists B \in \text{P and things } a_1, a_2, \ldots, a_n, \ldots \]

will

\[ 1 \leq |a_n| \leq n^c + n_0 \]

\[ x \in A \iff (x, a_{1x}) \in B. \]

Basic Containment

- \( \text{Time}(t) \subseteq \text{NTIME}(t) \subseteq \text{Time}(2^t) \)
- \( \text{Time}(t) \subseteq \text{Space}(t) \subseteq \text{Time}(2^t) \)

Slight improvements known for TM specific classes.

- \( \text{Time}(t) = \text{NTIME}(t) \) [Paul, Friggens, Sørensen, Trotter]
- \( \text{Time}(t) \leq \text{Space}(t/\log n) \)
Reductions & Completeness

- \( A \leq^* B \)
  - type: Turing / Karp
  - various possibilities
  - resource bound
  - poly time / log space

- "Usually" reduction composite

\[
A \leq_{\text{poly}}^\text{poly} B \leq_{\text{poly}} C
\]

\( \implies A \leq_{\text{poly}} C \)

Similarly with log space.

- Caution: \( A \leq_{\exp}^\exp B \leq_{\exp} C \)
  \( \not\implies A \leq_{\exp} C \).

Completeness: \( A \) is NP-complete if \( A \leq_{\text{NP}}^\text{NP} B \) \( \forall B \in \text{NP} \) \( B \leq_{\text{Poly}}^\text{Poly} A \)

"\( A \) is a hardest problem in NP".
Proving NP-Completeness

Option 1: Start from scratch. [Adopted by Cook/Karp]

- Start with formal definition of "B \in NP"
- Show how to reduce B to A

Applied to

[Applied to "SAT" = Input = Circuit
Goal = decide if Circuit satisfies ""]

Q: What description of B should we use?

Exercise: Given a circuit C that takes two inputs B - a description of circuit of size n
X - input of length n
\rightarrow Outputs B(X)

\[ B(x) \]
\[ C \]
\[ X \]
\[ B \]
Option 2: Recommended [Patented by Karp]
- Start with NP-complete problem B
- Show $B \leq^\text{poly} A$

**Claim:**

$B \leq^\text{poly} A \Rightarrow B \leq^\text{NP} A \Rightarrow A \text{ NP-complete}$

**Proof:**

$\forall B' \in \text{NP} \quad B' \leq^\text{poly} B \leq^\text{poly} A \Rightarrow B' \leq^\text{A}.\quad \uparrow \quad \text{assumption}$

$\text{NP-completeness of } B$

**Example:**

SAT $\leq 3\text{SAT} \quad [\text{Cook}]

Constraints:

\[ Z_4 = Z_2 \text{ or } Z_3 \]
\[ (Z_4 \lor \overline{Z}_2)(Z_4 \lor \overline{Z}_3) \]
\[ (Z_2 \lor \overline{Z}_3 \lor \overline{Z}_4) \]

[Aside: power & Non-determinism: Circuit $\Rightarrow$ depth 2 formula]
NP-Complete Problems: 3-Color, Hamiltonicity, TSP, CLIQUE,
Subset Sum, Integer Programming,

NL-Complete Problems

A \leq B
\log \text{space}

Key difference.

NL-Complete Problem

S-t connectivity in directed graphs: (ST-Conn)

Input: \( G = (V, E) \), \( S, T \in V \)

Goal: Decide if \( \exists \) path from \( S \) to \( T \) in \( G \).

Closely related

UST-Conn: Undirected conn. in undirected graphs

1. Savitch: UST-Conn, ST-Conn \( \in \log^2 n \) space \( [L^2] \)
2. Aleliunas, Karp, Lipton, Lovasz, Rackoff: UST-Conn \( \in \text{Randomized L} \)
3. Reingold
Pudding Theory

1. $P = NP \Rightarrow EXP = NEXP$

2. $L = NL \Rightarrow L^2 = NL^2$

Proof:

$A \in NEXP \Leftrightarrow A' = \{x \cdot 0^{ix} | x \in A\}$

$A' \in NP \Rightarrow A' \in P$

$\Rightarrow A \in EXP$

$NL \subseteq L^2$ [Switch] $\Rightarrow NSPACE(s) \subseteq SPACE(s^2)$

$\forall s(n) \geq \log n.$

Next lecture $NL \neq coNL$