CS 221 LECTURE 4 2/1/2018

TODAY:

- Space Complexity (quick defn.)
- Non-Determinism
  - NP
  - NEXP ?
  - NPSPACE
  - NL
- Complete Problems
- NP vs. CoNP vs. P
- NL vs. CoNL vs. L

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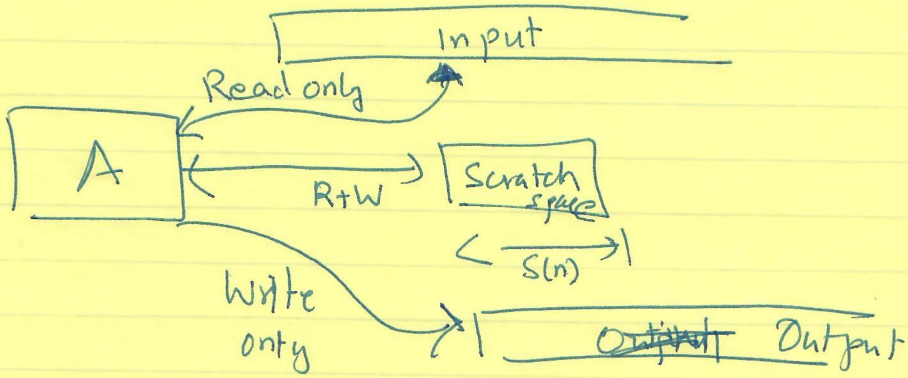
(1) Space Complexity of L.

Algorithm :
- gets to (re) read input.
- limited memory $S(n)$ bits.
- Outputs 0/1 answer [longer output ? Can write on output tape ...]



(2) Exercise / Pset 1:

if f_1 computable in space S_1

f_2 in " S_2

$\Rightarrow$ $f_1 \circ f_2$ in space $S_1 + S_2 + O(1)$.

Non-determinism:

- At ~~any moment~~ non-det. steps algorithm can make a choice.

- Q: $\exists$ sequence of choices that makes algorithm accept.

Leads to Non-det. time NSPACE ...

Alt / Equiv view of NP

- "Pair Languages" : ~~Input~~ Instances are pairs $\in \{0,1\}^* \times \{0,1\}^*$

- $A \in NP$ if $\exists$ pair language $B \in P$ & c, n_0 s.t. $x \in A \iff \exists y$ s.t.
$$(x,y) \in B$$
$$|y| \leq |x|^c + n_0$$

- $L \in NEXP$? What changes?

Alt/Equiv view of NL

- $A \in NL$ if $\exists$ ~~logspace~~ a algorithm $B(x,y)$,
  - that accesses "y" only left to right,
  - uses $O(\log(|x|))$ space

  & $x \in A \iff \exists y$ s.t. $B(x,y) = 1$.

Aside: Quantifiers & Pair Languages

· Very useful notation; Good to start thinking in such terms.

Example

$A \in P/_{poly}$ if $\exists$ $\overset{n_0, c}{B \in P}$ & strings $a_1, a_2, \ldots a_n, \ldots$

with

① $|a_n| \leq n^c + n_0$

② $x \in A \iff (x, a_{|x|}) \in B$.

Basic Containments

· $\text{Time}(t) \subseteq \text{NTIME}(t) \subseteq \text{TIME}(2^t)$

· $\text{Time}(t) \subseteq \text{Space}(t) \subseteq \text{Time}(2^t)$.

Slight improvements known for TM specific classes.

OMITTED

① $\text{Time}(t) \neq \text{NTIME}(t)$. [Paul-Pippenger-Szemeredi-Trotter]

② $\text{Time}(t) \subseteq \text{Space}(t/\log n)$

Reductions & Completeness

- $A \leq_{*} B$ iff

  $\uparrow$ various possibilities $\xrightarrow{\text{type}}$ Turing / Karp

  $\searrow$ resource bound: poly time / log space ...

- "Usually" reductions compose

$$A \leq_{poly} B \quad \& \quad B \leq_{poly} C$$

$$\Rightarrow \quad A \leq_{poly} C$$

similarly with log space.

- Caveat: $A \leq_{exp} B \quad \& \quad B \leq_{exp} C$

$$\not\Rightarrow \quad A \leq_{exp} C.$$

Completeness: A is NP-Complete if $A \in NP$ &

$$\forall B \in NP \qquad B \leq_{poly} A$$

"A is a hardest problem in NP".

Proving NP-Completeness

Option 1: Start from scratch. [Adopted by Cook/Levin]

- Start with formal definition of "$B \in NP$"
- Show how to reduce B to A.

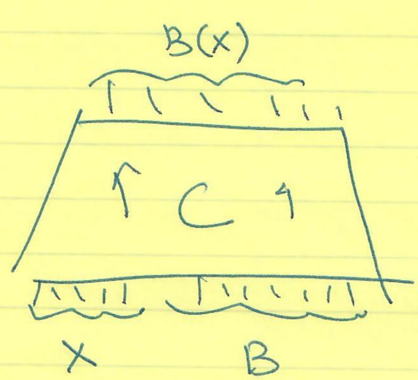
~~Applied to~~

[Applied to "SAT" = Input = Circuit
Goal = decide if Circuit satisfiable "]

Q: What description of B should we use?

Exercise: Given a circuit C that takes two inputs B – a description of circuit of size n
X – input of length n

▷ Outputs $B(X)$.



Option 2 : Recommended [Patented by Karp]

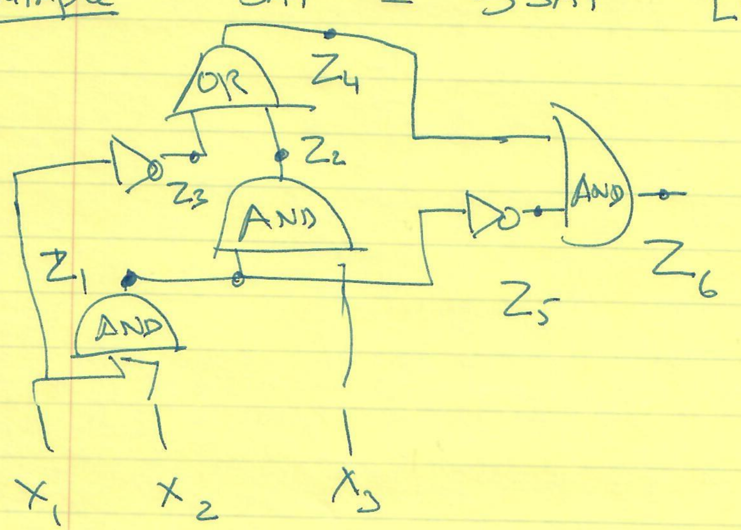
- Start with NP-complete problem B
- & show $B \leq_{poly} A$

Claim: $B \leq_{poly} A$ & B NP-complete & $A \in NP$

$\Rightarrow$ A NP-complete.

Proof. $\forall B' \in NP$ $\quad B' \underbrace{\leq_{poly}}_{\text{NP-Completeness of } B} B \underbrace{\leq_{poly}}_{\text{assumption}} A \Rightarrow B' \leq A.$

Example $SAT \leq 3SAT$ [Cook]



Constraints

"$Z_4 = Z_2$ OR Z_3"

$\Downarrow$

$(Z_4 \vee \bar{Z}_2)(Z_4 \vee \bar{Z}_3)$

$(Z_2 \vee Z_3 \vee \bar{Z}_4)$

[Aside power of Non-determin : Circuit $\Rightarrow$ depth 2 formula]

Other
NP-Complete Problems : 3-Col. , Hamiltonicity , TSP, Clique,
Subset Sum, Integer Programming,

NL - Completeness

$$A \le_{\text{logspace}} B$$

↑
key difference.

NL-Complete Problem

s-t connectivity in directed graphs. (ST-Conn)

Input : $G = (V, E)$; $s, t \in V$
Goal : Decide if $\exists$ path from s to t in G.

Closely related

UST-Conn : ~~Undirected~~ s-t conn. in undirected graphs

(1) Savitch : UST Conn, ST Conn $\in$ $\log^2 n$ space $[L^2]$

(2) Aleliunas-Karp-Lipton-Lovasz-Rackoff : USTconn $\in$ RandomizedL

(3) Reingold $\in L$.

Padding Theory

- $P = NP \Rightarrow EXP = NEXP$
- $L = NL \Rightarrow L^2 = NL^2$

Proof: ~~$A \in P \Rightarrow$~~

$A \in NEXP \Leftarrow A' = \{x \cdot 0^{2^{|x|}} \mid x \in A\}$

$A' \in NP \Rightarrow A' \in P$

$\Rightarrow A \in EXP$ $\boxtimes$

$NL \subseteq L^2$ [Savitch] $\Rightarrow NSPACE(s) \subseteq DSPACE(s^2)$

$\forall s(n) \geq \log n.$

Next lecture $NL = coNL$ ...