Today

SPACE & NON-DETERMINISM

- $L$ vs $NL$ vs. $coNL$

- $PSPACE = NPSPACE = coNPSPACE$

- $PSPACE$, Games, $QBF$

--- x ---

Review of Time vs. Space vs. Non-Determinism

$(A \rightarrow B \text{ below } \Rightarrow A \leq B)$

--- x ---

A $\rightarrow$ B (To be proved today)
Understanding NL

1. ST-conn (aka PATH) is NL-complete

   Input: Directed graph $G$, vertices $s,t$.
   Goal: Find a path from $s$ to $t$ in $G$.

   [Proof: Exercise]

2. ⊗ PSI Problem 4: $A^k$ can be computed in space $O(k \cdot \log n)$

   (i) $(A^n)_{s,t} = 1 \iff$ Graph with adjacency matrix $A$ has a path from $s \rightarrow t$ (assume $A_{ss} = 1$)

   (ii) $NL \subseteq SPACE (\log^2 n) \subseteq L^2$

   $NPSPACE = PSPACE = \bigotimes NPSPACE$

Aside: $NL \subseteq L^2 \cap \#P$

Open: $NL \subseteq TIME \cdot SPACE (\text{poly}(n), \text{polylog}(n))$?

"Steve's Claim"
[l for Stephen Cook]
Main Theorem

Key Result: Immerman-Szelepsenyi '88

Theorem: $\text{NL} = \text{CoNL}$

Proof:

1. Will give non-deterministic logspace algorithm that verifies there is no path from $S$ to $T$ in $G_1$.

2. Will give deterministic logspace algorithm $V(G_1, S, T, k, N_{k-1}; \pi)$ with the following property

   (1) If $N_{k-1} = \#$ vertices reachable by paths of length $\leq k-1$ from $S$ in $G_1$

   $\exists \pi \text{ path of length } \leq k \text{ from } S \text{ to } T$

   then $\exists \pi \pi = \pi \sigma, \sigma \in \Sigma^* V(, \pi) = 1$

   (2) If $N_{k-1} = \#$ vertices reachable by paths of length $\leq k-1$ from $S$ in $G_1$

   $\forall \text{ no path of length } \leq k \text{ from } S \text{ to } T$

   then $\forall \pi \exists \pi \pi \neq \pi V(, \pi) = 0$. 


• Will defer construction $\mathcal{V}(\ )$.

• But why is it useful? How to get $N_r$?

• Key idea: **Inductive-Counting**

  Can we $N_{r-1}$ to (non-deterministically) compute $N_r$?

$\mathcal{W}(G, s, a, N_{r-1}, N_r; \psi)$:

Output:

• if $N_{r-1} = \#$ vertices in $G$ reachable in $\leq k-1$ steps from $s$

  1. $N_r = \... \leq k \...

  then $\exists \psi \text{ s.t. } \mathcal{W}(\psi) = 1$

• if $N_{r-1} = \text{correct}$

  2. $N_r = \text{incorrect}$

  then $\forall \psi \mathcal{W}(\psi) = 0$
\[ \Psi = (\langle i, b_i, \prod_{i,k}^{b_i} \rangle_{i \in V}) \]

\[ b_i = 1 \text{ if } \exists \text{ path from } S \text{ to } i \text{ of length } \leq k \]

\[ \prod_{i,k}^1 = V_1, V_2, \ldots, V_k = i \quad \text{and } S \rightarrow V_1 \rightarrow V_2 \cdots \rightarrow V_k \]

(proof: \exists \text{ path from } S \rightarrow i \text{ of length } \leq k)

\[ \prod_{i,k}^0 = \prod_{i,k}^1 \quad \text{ s.t. } \quad V(6_1, s, i, k, N_{R-1}; \prod_{i,k}^1) = 1 \]

(proof: no path from \( S \rightarrow i \) of length \( \leq k \))

\[ W(6_1, s, b, N_{R-1}, N_k; \Psi) : \]

Accept if \( \sum_{i=1}^{n} b_i = N_k \) and

\[ \forall i \quad V(6_1, s, i, k, N_{R-1}; \prod_{i,k}^1) = 1 \quad \text{and } \quad b_i = 0 \]

or \( S \rightarrow V_1 \rightarrow \cdots V_k = i \) \quad and \quad \( b_i = 1 \)

\[ \text{PATH}(6_1, s, t; \Psi_1, \Psi_2, \ldots, \Psi_n, N_0 = 1) \]

for \( r = 1 \) to \( n \) do

if \( W(6_1, s, k, N_{R-1}, N_k; \Psi) = 0 \) reject

else continue

Accept if \( V(6_1, s, t, N_k; \prod_{t,n}) = 1 \)
Algorithm for $V$

To prove no path of length $\leq k$ from $s$ to $t$ given $N_{r-1}$

1. Given $N = N_{r-1}$ vertices $V_1 \ldots V_N$ with proofs that
   a. $s \to V_1$ has path of length $\leq k-1$
   b. $V_i \not\to t$.

$$V(G,s,t,k,N_{r-1}; \Pi)$$

"$\Pi = (i, b_i, \langle V_{1} \ldots V_{r-1} = i \rangle)_{i=1}^{n}$"

Accept if $\sum b_i = N_{r-1}$

- for every $i$ either $b_i = 0$
- or $s \to V_i \to \ldots V_{r-1} \to i$

2. $i \not\to t$. 
**PSPACE = Complexity of Games**

**zero-sum**

**Game \(n\)-move game**: Initial state = \(n\) bits = \(x\)

- \(n\) alternating moves \(P_1 \& P_2\)
  - \(x\) given \((x, m_1, m_2, \ldots, m_n)\)
  - Decidable in \(P\) if \(P_1\) wins.
  - \([P_2\ \text{wins} \iff P_1\ \text{does not}]\)

(Say by alg/circuit \(C(x, m_1, m_2, \ldots, m_n)\))

\[x \in L_{\text{game}} \iff \exists m_1 \forall m_2 \ldots \exists m_n \forall m_n \ C(x, m_1, m_2, \ldots, m_n) = 1\]

**Claim**: \(L_{\text{game}} \subseteq \text{PSPACE}\)

**Proof**: Enumerate all \(m_1, \ldots, m_n\) ....

**Claim 2**: \(\text{PSPACE} \subseteq L_{\text{game}}\)

**Key Idea**: (Same as Shvitz's Thm aka \(A^k \in \text{Space}(logk)\))

**Player 1**: "\(\exists\) accepting path from \(S \rightarrow T\) in \(2^n\) steps"

Furthermore \(M_i = S_{2^{n-1}}\) is midpoint;

**Player 2**: No path from \(S \rightarrow S_{2^{n-1}}\ or \ S_{2^{n-1}} \rightarrow T\)
in \(2^{n+1}\) steps in \(2^n\) steps.
Define: 

\[ \text{QBF} \left\{ \phi \right\} : \exists x_1 \land \exists x_2 \land \exists x_3 \cdots \land \exists x_n \phi(x_1, \ldots, x_n) = 1 \]

in 3CNF

Theorem: QBF is \text{PSPACE}-Complete.

- Alternations = \text{PSPACE} (quantifiers)
- 1 quantifier = \text{NP} (coNP)
- 2 quantifiers = ?

AT\text{PSPACE} (a(n), t(n), s(n)) = ?

\[ \text{Next lecture.} \]