CS 221 Lecture 5 2/6/2018

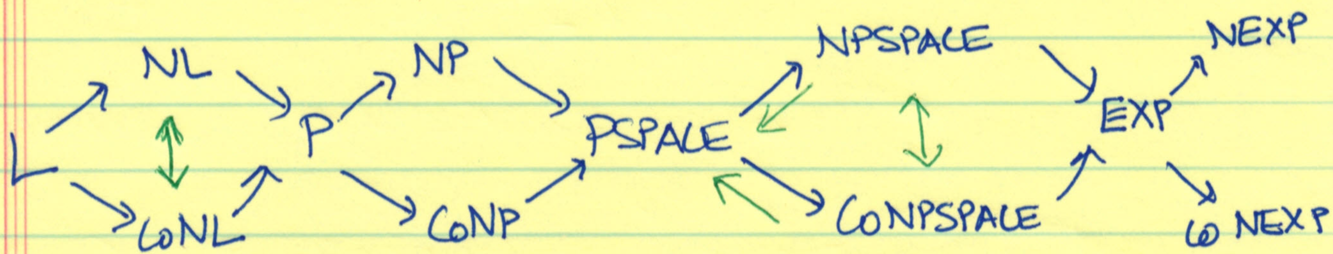
Today

Space & Non-Determinism

- L vs. NL vs. coNL;
- PSPACE = NPSPACE = coNPSPACE;
- PSPACE, Games, QBF

Review of Time vs. Space vs. Non-Determinism

($A \to B$ below $\Rightarrow$ $A \subseteq B$)



$A \to B$ (To be proved today)

Understanding NL

1. S-T-CONN (aka PATH) is NL-Complete

Input: Directed Graph G, vertices s,t.
Goal: $\exists$ path from s to t in G.

[Proof: Exercise]

2. (i) PS1 Problem 4: A^k can be computed in space $O(\log k \cdot \log n)$

(ii) $(A^n)_{s,t} = 1 \iff$ Graph with adjacency matrix A has Path from $s \to t$

(assume $A_{ii} = 1$)

(iii) $NL \subseteq SPACE(\log^2 n) \triangleq L^2$

$NPSPACE = PSPACE = coNPSPACE$

Aside: $NL \subseteq L^2 \cap P$

Open: $NL \subseteq$ TIME-SPACE(poly(n), polylog(n))?

"Steve's Class"
[for Stephen Cook]

Main Theorem
~~Key Result~~: Immerman - Szelepcsenyi '88

Theorem : NL = CoNL

Proof: • Will give Non-Deterministic logSPACE algorithm that verifies there is no path from s to t in G.

• ≡ Will give Deterministic Logspace algorithm $V(G, s, t, k, N_{k-1}\,;\, \Pi)$ with the following property

① if N_{k-1} = # vertices reachable by paths of length $\le k-1$ from s in G

& $\exists$ path of length $\le k$ from s to t

then $\exists\, \Pi = \overline{\Pi}_{t,k}$ s.t. $V(\quad ; \Pi) = 1$

② if N_{k-1} = # vertices reachable by paths of length $\le k-1$ from s in G

& No path of length $\le k$ from s to t

then $\forall$ ~~no~~ Π ~~s.t.~~ $V(\quad ; \Pi) = 0$.

- Will Defer construction of $V(\quad)$.
- But Why is it useful? How to get N_k?
- Key Idea: <u>INDUCTIVE-COUNTING</u>

Can use N_{k-1} to (non-deterministically) compute N_k.

$W(G, s, N_{k-1}, N_k ; \psi)$:

~~Outputs~~

- if N_{k-1} = # vertices in G reachable in $\leq k-1$ steps from s
  & N_k = " $\leq k$ "

  then $\exists \psi$ s.t. $W(\quad ; \psi) = 1$

- if N_{k-1} = correct
  & N_k = incorrect

  then $\forall \psi$ $W(\quad ; \psi) = 0$

$$\psi = \left(\langle i, b_i, \Pi_{i,k}^{b_i} \rangle_{i \in V}^{*} \right)$$

$b_i = 1$ if $\exists$ path from s to i of length $\leq k$

$\Pi_{i,k}^{1} = v_1, v_2 \ldots v_k = i$ & $s \to v_1 \to v_2 \ldots \to v_k$

(proof $\exists$ path from $s \to i$ of length $\leq k$)

$\Pi_{i,k}^{0} = \overline{\Pi}_{i,k}$ s.t. $V(G, s, i, k, N_{k-1} ; \overline{\Pi}_{i,k}) = 1$

(proof no path from $s \to i$ of length $\leq k$)

$$\boxed{\begin{array}{l} W(G, s, k, N_{k-1}, N_k ; \psi) : \\ \quad \text{Accept if } \sum_{i=1}^{n} b_i = N_k \ \& \\ \quad \forall i \quad V(G, s, i, k, N_{k-1} ; \overline{\Pi}_{i,k}) = 1 \ \& \ b_i = 0 \\ \quad \text{or} \quad s \to v_1 \to \ldots v_k = i \quad \& \ b_i = 1 \end{array}}$$

$\overline{\text{PATH}}(G, s, t ; \cancel{\psi^1, \psi^2, \ldots \psi^n, N} \ (N_1, \psi^1), (N_2, \psi^2) \ldots (N_n, \psi^n), \Pi_{t,n})$

$N_0 = 1$

for $k = 1$ to n do

  if $W(G, s, k, N_{k-1}, N_k ; \psi) = 0$ reject

  else continue

Accept if $V(G, s, t, n ; N_n ; \Pi_{t,n}) = 1$

Algorithm for V

To prove <u>no</u> path of length $\leq k$ from s to t given N_{k-1}

1. Given $N = N_{k-1}$ vertices $v_1 \dots v_N$ with proofs that

   a. $s \rightsquigarrow v_i$ has path of length $\leq k-1$

   b. $v_i \not\rightarrow t$.

$$V(G, s, t, k, N_{k-1} ; \pi)$$

$$\text{"} \pi = \left(i, b_i, \langle v_1 \dots v_{k-1} = i \rangle \right)_{i=1}^{n} \text{"}$$

Accept if $\Sigma b_i = N_{k-1}$

& for every i either $b_i = 0$

or $s \to v_1 \to \dots v_{k-1} \to i$

& $i \not\to t$.

PSPACE = Complexity of Games

~~2 player~~ Game - n-move zero-sum game : • Initial State = n bits = X

• n - alternating moves P_1 & P_2

• ~~X~~ Given $(X, m_1, m_2, \ldots m_n)$

decidable in P if P_1 wins.

[P_2 wins $\Leftrightarrow$ P_1 does not]

(say by alg/ circuit $C(x, \ldots m_1 \ldots m_n)$)

$$X \in L_{game} \Leftrightarrow \exists m_1 \forall m_2 \cdots \exists m_{n-1} \forall m_n \; C(X, m_1, m_2 \ldots m_{n}) = 1$$

• Claim : $L_{game} \in$ PSPACE

Proof : Enumerate all $m_1 \ldots m_n \ldots$ $\boxtimes$

• Claim 2 : PSPACE $\leq L_{game}$.

Key Idea : (same as Savitch's Thm aka $A^k \in \mathrm{Space}(\log k \log n)$)

Player 1 : "$\exists$ accepting path from $S \to t$ in 2^n steps"

Furthermore $m_1 = S_{2^{n-1}}$ is midpoint ;

Player 2 : No path from $S \to S_{2^{n-1}}$ in 2^{n-1} steps or $S_{2^{n-1}} \to t$ in 2^{n-1} steps.

$s \to t$ in 2^n

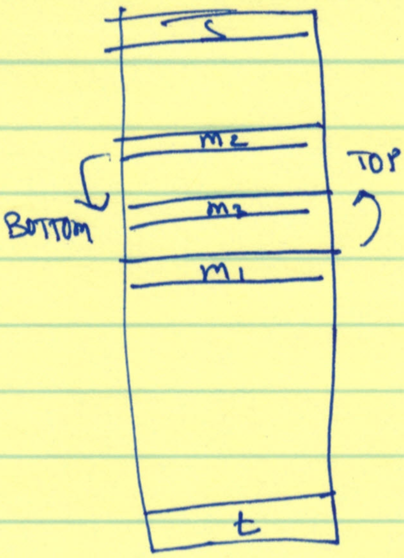
P_1

$\xrightarrow{\quad m_1 \quad}$

$\xleftarrow{\text{TOP / BOTTOM}}$

$\xrightarrow{\quad m_2 \quad}$

$\xleftarrow{\text{TOP / BOTTOM}}$



$\xrightarrow{\qquad\qquad}$

at end $S_j \to S_{j+1}$ in 1 step $\Rightarrow$ can be verified in poly time.

Def: ~~Thm~~: QBF $\{\phi : \exists x_1 \forall x_2 \exists x_3 \dots \forall x_n \, \phi(x_1 .. x_n) = 1\}$

3CNF

Thm: QBF is PSPACE-Complete.

n - alternations (quantifiers) = PSPACE

1 quantifier = NP (coNP)

2 quantifier = ?

ATISP $(a(n), t(n), s(n))$ = ?

Next lecture.