1. Sparse Problems.
   A sparse problem is one where, for each $n \in \mathbb{N}$, the number of accepting inputs of length $n$ is bounded by a polynomial in $n$.
   
   Show that if a sparse problem is NP-complete, then the polynomial hierarchy collapses to the second level (i.e., $PH = NP$).
   
   For extra credit (i.e., optional, but no discussions/collaborations/scouring-the-web allowed): Show that if a sparse problem is NP-complete then $NP = P$.

2. Search BPP.
   Recall the notion of promise-BPP-search alluded to in Lecture 8. Roughly, promise-BPP-search is the class of problems with probabilistic search, and verification routines. Formally, the following promise problem is complete for promise-BPP-search:
   
   - **Input:** A “search” circuit $C : \{0,1\}^k \to \{0,1\}^n$ and a verification circuit $V : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}$ (the search circuit searches for a solution, and the verification verifies a solution, both probabilistically).
   
   - **Promises:**
     
     $$\forall x \in \{0,1\}^n, \Pr_y[V(x, y) = 1] \notin (1/3, 2/3)$$
     
     and
     
     $$\Pr_z \left[ \Pr_y[V(C(z), y) = 1] \geq 2/3 \right] \geq 2/3$$
   
   - **Task:** Output $x$ such that $\Pr_y[V(x, y) = 1] \geq 2/3$.
     
     (a) With the above definition, show that promise-BPP-search can be amplified, so that $\Pr_y[V(x, y) = 1] \notin (2^{-\Omega(n)}, 1 - 2^{-\Omega(n)})$ and $\Pr_z \left[ \Pr_y[V(C(z), y) = 1] \geq 1 - 2^{-\Omega(n)} \right] \geq 1 - 2^{-\Omega(n)}$.
     
     (b) Show that if promise-BPP = P, then promise-BPP-search = P. (Hint: Find $z$ such that $C(z)$ is a solution to the search problem. Try to find $z$ one bit at a time by defining appropriate promise-BPP problems.)

3. Relaxed BPL
   Give a formal definition of BPL, the class of problems that can be solved with randomized logspace algorithms and polynomial running time.

   The rest of this question - motivates the explicit running time restriction. Consider a relaxation of BPL where we do not restrict the runtime of the machine to be polynomial. Show that Relaxed-BPL contains NL.

4. Derandomizing BPL
   Show that $BPL$ is in $L^2$.
   
   (Hint: Consider the technique used in Problem 4 on Pset1, to show $NL \subseteq L^2$)