ITCS: Harvard CS 229r 3/26/2014

LECTURE 15

TODAY:

COMPRESSING INTERACTIVE COMMUNICATION

Recall: Given inputs $X \in \{0,1\}^n$ & $Y \in \{0,1\}^n$ to Alice + Bob respectively, a protocol $\Pi = (\Pi_1, \dots \Pi_r)$ is generated using shared randomness R & private randomness R_A, R_B respect. as

$$\Pi_1 = \Pi_1(X, R, R_A)$$

$$\Pi_2 = \Pi_2(Y, R, R_B ; \Pi_1)$$

$$\vdots$$

$$\Pi_i = \Pi_i(X, R, R_A ; \Pi_{<i}) \quad \text{if } i \text{ odd}$$

$$= \Pi_i(Y, R, R_A ; \Pi_{<i}) \quad \text{if } i \text{ even}$$

$$\vdots$$

$$\Pi_r = \Pi_r(\; \dots \;).$$

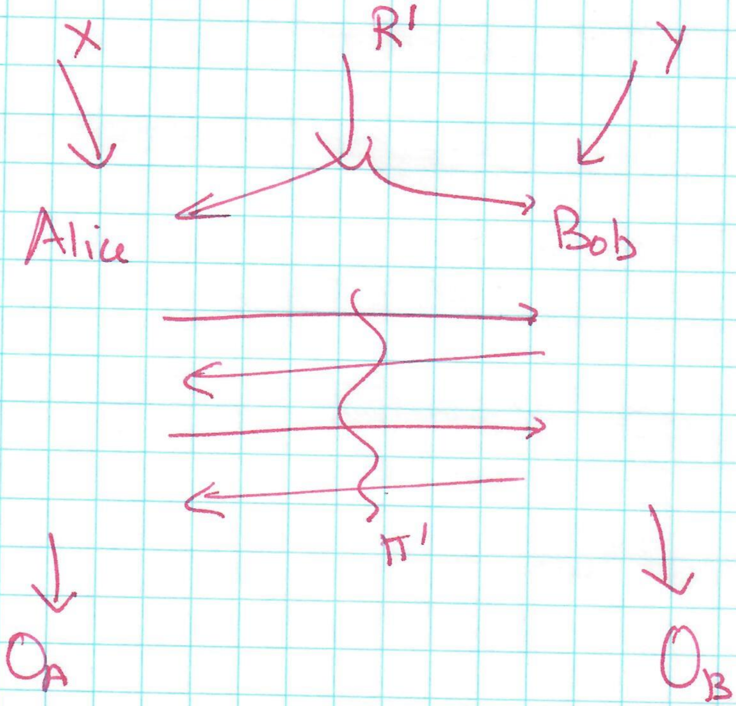
Communication cost $CC(\Pi) = |\Pi_1| + \dots |\Pi_r| \geq H(\Pi_1) + \dots H(\Pi_r)$.

[assuming $(X,Y) \sim \mu$].

Information Cost

$$IC_\mu^{int}(\Pi) \triangleq \underbrace{I(X ; \Pi \mid Y, R) + I(Y ; \Pi \mid X, R)}_{\text{Information learned about } X \,\&\, Y \text{ by Alice \& Bob from interaction}}$$

Defn π' simulates π if



$O_A = (R, \pi) \quad = \quad O_B = (R, \pi)$

& (R, π, X, Y) distributed exactly as in π.

Theorem [Barak, Braverman, Chen, Rao]: $\forall\, \pi$ s.t. $IC(\pi) = I$ & $CC(\pi) = C$ $\exists\, \pi'$ s.t. $CC(\pi') \leq I\,C\,\mathrm{polylog}\, C$ & π' simulates π.

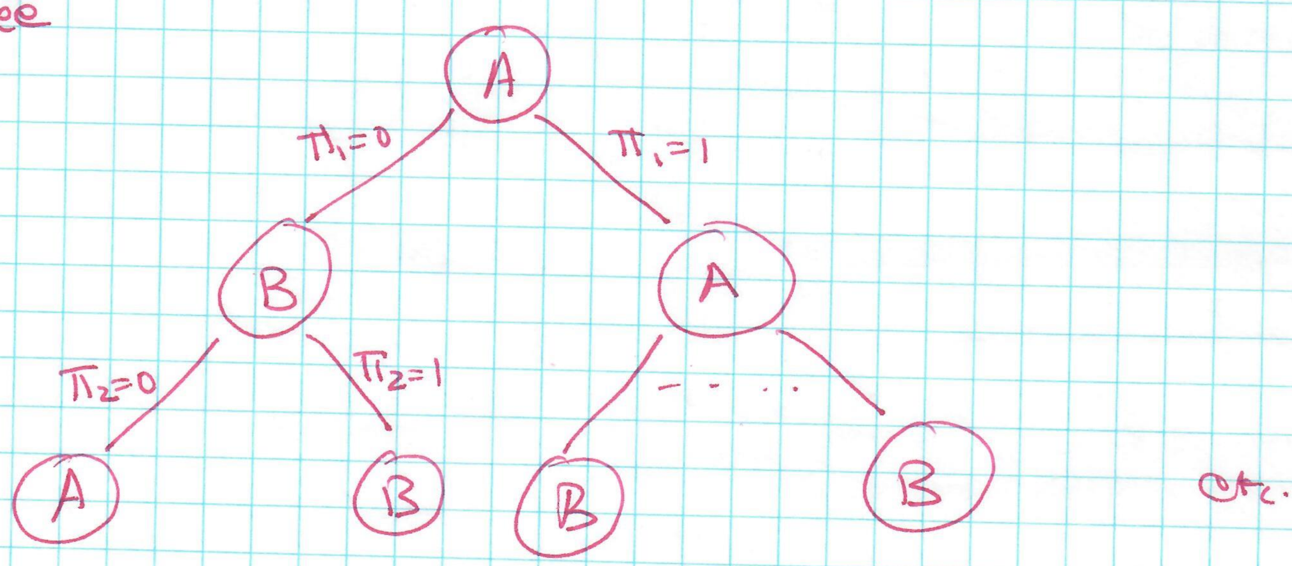
"Can compress interactions if information cost is low".

Motivation: Direct Sum / Amortized Cost. Will see next lecture.

Proof: from now on.

Protocol Trees, Priors, Information Cost

- Let Π be protocol we want to simulate. Π' = Simulating protocol
- Wlog Π has no public randomness
- $\Pi = (\Pi_1, \Pi_2, \ldots \Pi_k)$ $\quad k = CC(\pi)$
- Wlog $\Pi_i \in \{0,1\}$
- Tree



- Deterministic Protocol $\Rightarrow$ "owner" knows which path to take
- Prob. (Private Randomness) Protocol $\Rightarrow$ "owner" knows Prob of going left/right.
- IC low $\Rightarrow$ other player also has good idea.

$$IC_\mu(\Pi) = I(X;\Pi|Y) + I(Y;\Pi|X)$$

$$= \sum_{i=1}^{k} \underbrace{I(\Pi_i ; X | Y, \Pi_{<i})} + \underbrace{I(\Pi_i ; Y | X, \Pi_{<i})}$$

- one of these always zero. Why?
- use this to prove $\underline{IC}^{int} \leq \underline{IC}^{ext}$.

$$= \sum_{i=1}^{k} V_i \qquad \left(V_i \triangleq I(\Pi_i ; X | Y, \Pi_{<i}) + I(\Pi_i ; Y | X, \Pi_{<i}) \right)$$

$IC(\Pi) << CC(\Pi) \Rightarrow$ typical V_i very small.

Why does this help compress.

Extreme Case (for Intuition): $V_i = 0 \quad \forall i$.

$\Rightarrow$ Alice & Bob <u>both</u> know prob. going left at each node.

$\Rightarrow$ Can sample appropriate path / leaf using <u>common</u> randomness. (Note Π' can use common randomness).

$\Rightarrow$ leads to zero comm protocol that simulates Π perfectly (gets same distribution).

Next: IC very small (example);

- Suppose Alice owns all nodes.
- Bob thinks all prob are $(\frac{1}{2}, \frac{1}{2})$
- Alice's actual probs are $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$.
- final distribution of leaves at TVD $\leq k \cdot \delta$ from uniform.
- Task: Sample leaves according to near-correct distribution (with zero communication).

  Allow Alice + Bob to have different distributions with some probability

Correlated Sampling:

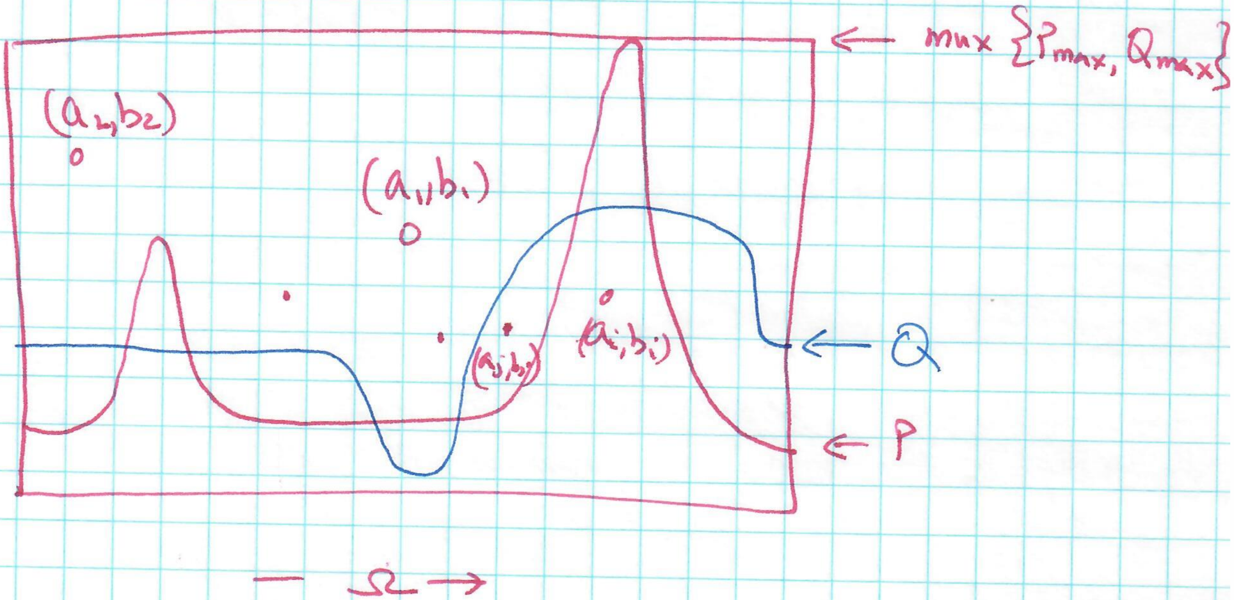
$$\begin{tikzcd}
 & & R \arrow[dl, bend right] \arrow[dr, bend left] & & \\
\text{Alice} \arrow[d] & P \arrow[l] & (P,Q \text{ distributions on } \Omega) & Q \arrow[r] & \text{Bob} \arrow[d] \\
w_A & & & & w_B
\end{tikzcd}$$

Want: - $w_A \sim P$, $w_B \sim Q$

- $\Pr[w_A \neq w_B]$ small

Solution : [Broder, Kleinberg-Tardos, Holenstein ...]

"Dart-Throwing"



- Draw Prob. Distributions as above
- Common randomness = random sequence of points in rectangle $(a_1,b_1), (a_2,b_2) \ldots$
- Alice : Picks first point (a_i,b_i) under P curve ; Outputs a_i.
- Bob : Same ... Q - curve.

Lemma: $\Pr[W_A \neq W_B] \leq 1 - \dfrac{1 - TVD(P,Q)}{1 + TVD(P,Q)} \leq 2\,TVD(P,Q)$

Proof: Exercise.

Apply to simple example

- $\Pr[\text{Alice's leaf} \neq \text{Bob's Leaf}] \leq k\delta$.
- $IC \stackrel{?}{=} k \cdot \underbrace{\delta^2}_{} = \epsilon$

  ↑ Divergence(Bern($\frac{1}{2}$) || Bern ($\frac{1}{2}-\delta$))

- Error Prob. $= k\delta = \sqrt{k} \cdot \sqrt{k\delta^2} = \sqrt{C} \cdot \sqrt{I}$!

  [Need to convert this to an $\sqrt{IC}$ comm. protocol with negligible error!]

BBCR Protocol

Setup: for every node u on protocol tree

let P_v^A – denote Alice's guess on prob going right

P_v^B – denote Bob's "

P_v ← denote "owner"'s prob. of going right.

$[\; P_v^A \sim (\pi_i \mid \pi_{<i}, X) \;;\; P_v^B - (\pi_i \mid \pi_{<i}, Y) \;;\; P_v - (\pi_i \mid \pi_{<i}, XY) \;]$

Goal: Sample path from root to leaf according to P

Protocol

- Alice samples leaf according to p^A
- Bob (correlatedly) samples according to p^B
- Perform binary search to find LCA, u.
- Repeat from u

Analysis

- Comm. per iteration $\doteq O(\log C)$
- # iterations = ?

$Z_i = 1$ if disagreement on i^{th} level

iterations $= \mathbb{E}\left[\sum Z_i\right]$.

- $\mathbb{E}[Z_i] = \mathrm{TVD}\left((\Pi_i \mid X, \Pi_{<i}), (\Pi_i \mid Y, \Pi_{<i})\right)$
- $V_i \;(= I(\Pi_i ; X \mid \Pi_{<i} Y) + I(\Pi_i ; Y \mid \Pi_{<i}, X))$

$$= D\left((\Pi_i \mid X, \Pi_{<i}) \,\|\, (\Pi_i \mid Y, \Pi_{<i})\right)$$

- By Pinsker, $\mathbb{E}[Z_i] \leq \sqrt{V_i}$

$$\Rightarrow \mathbb{E}\left[\sum Z_i\right] = \sum \sqrt{V_i} \leq \sqrt{k} \cdot \sqrt{\sum V_i} = \sqrt{IC}.$$

$\boxtimes$