**CS 229r Information Theory in Computer Science** Apr 23, 2019

# Lecture 23

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## 1 Administrative Notes

- Project Presentations on Wed 5/1, emphasize one interesting point if do not have enough time
- Writeup ($\sim 5$ pages) due Wed 5/8
- Polished Scribe Notes (including worked out exercises) due Wed 5/8

## 2 Today's Agenda

- Amplification/Polarization of $SD$
- $SD \leq \overline{SD}$

## 3 Amplification/Polarization of SD

Recall that to define sampleable distributions, we define a circuit $C$ with $m$ inputs and $n$ outputs: $c : \{0,1\}^m \to \{0,1\}^n$ where $m, |c| = \text{poly}(n)$. Then for $X \sim \text{Bern}(\frac{1}{2})^m$, $C(X)$ defines a distribution on $\{0,1\}^n$, and we use $C$ to represent this distribution for simplicity of notations.

Given sampleable distributions and two parameters $c, f$ ($0 \leq c \leq f \leq 1$), we can define two sets:

$$\text{CLOSE}^c = \{(c_1, c_2) | \delta(c_1, c_2) \leq c\}$$
$$\text{FAR}^f = \{(c_1, c_2) | \delta(c_1, c_2) \geq f\}$$

where $\delta(P, Q)$ is the statistical difference between distributions $P$ and $Q$ (see last lecture for the definition of statistical difference).

**Definition 1** (Statistical Difference Problem)**.** Given $(c_1, c_2) \in \text{CLOSE}^c \cup \text{FAR}^f$, the statistical difference problem $\text{SD}^{c,f}$ is to decide whether $(c_1, c_2) \in \text{CLOSE}^c$ (return YES) or $(c_1, c_2) \in \text{FAR}^f$ (return NO).

For completeness of this definition, we can consider its complement problem.

**Definition 2** (Complement of the Statistical Difference Problem)**.** The complement of $\text{SD}^{c,f}$ is $\overline{\text{SD}}^{c,f}$, which returns NO if $(c_1, c_2) \in \text{CLOSE}^c$ and YES if $(c_1, c_2) \in \text{FAR}^f$ for $(c_1, c_2) \in \text{CLOSE}^c \cup \text{FAR}^f$.

As mentioned in the last lecture, we are interested in $\text{SD}^{\frac{1}{3}, \frac{2}{3}}$ due to its relation to the problem of Graph Isomorphism. For simplicity we omit the superscripts and call it SD. We want to ask if we can amplify this problem into $SD^{2^{-n^\varepsilon}, 1-2^{-n^\varepsilon}}$ which is more polarized since $c$ comes closer to 0 and $f$ comes closer to 1.

**Theorem 1.** *$SD^{\frac{1}{3}, \frac{2}{3}} \leq SD^{2^{-n^\varepsilon}, 1-2^{-n^\varepsilon}}$, where we can make $\varepsilon$ arbitrarily close to 1. More generally, the proof goes through as long as $c < f^2$.*

To prove Theorem 1, we need to use two kinds of reductions.

**Lemma 1** (The Direct Product reduction)**.** $SD^{c,f} \leq SD^{tc, 1-2\exp(-tf^2/2)}$

We begin with the direct product reduction because it's simpler to prove (during class this was called Ingredient 2). To prove Lemma 1, we simply map any $(c_1, c_2)$ to $(c_1^t, c_2^t)$, where

$$c^t(X_1, X_2, \cdots X_t) = (c(X_1), c(X_2), \cdots c(X_t))$$

To prove that the reduction finds a solution to $SD^{tc, 1-2\exp(-tf^2/2)}$, we need to prove that

(i) $\delta(c_1^t, c_2^t) \leq t\delta(c_1, c_2)$

(ii) for $\delta(c_1^t, c_2^t) \geq f$, $\delta(c_1^t, c_2^t) \geq 1 - 2\exp(-tf^2/2)$

To prove (i), notice that $\delta$ is a distance metric and we can apply triangular inequality multiple times, each time replacing one element in the sequence beginning with $c_1(X_1), \cdots c_1(X_t)$, until we arrive at $c_2(X_1), \cdots c_2(X_t)$.

To prove (ii), from the definition of statistical difference, because $c_1$ and $c_2$ are apart from each other by at least $f$, $\exists$ test $T$ and value $\alpha$ such that

$$\mathbb{E}_{z\sim c_1}[T(z)] \geq \alpha + f$$
$$\mathbb{E}_{z\sim c_2}[T(z)] \leq \alpha$$

Then our new test for $(c_1^t, c_2^t)$ returns 1 if $T(z_1) + T(z_2) + \cdots T(z_t) \geq (\alpha + \frac{f}{2})t$ else 0.

**Exercise 1.** *Prove that $\delta(c_1^t, c_2^t) \geq 1 - 2\exp(-tf^2/2)$ for $\delta(c_1^t, c_2^t) \geq f$ in a rigorous way.*

*Proof.* For the test defined above, we get 1 when $T(z_1) + T(z_2) + \cdots T(z_t) \geq (\alpha + \frac{f}{2})t$, or equivalently

$$\frac{1}{t}\sum_{i=1}^{t} T(z_i) \geq \alpha + \frac{f}{2}$$

If we sample from $c_1$, the mean of $T(z)$ for $z \sim c_1$ is $\alpha + f$, so applying the Chernoff Bound, we have w.p. at least $1 - \exp(-tf^2/2)$m

$$\frac{1}{t}\sum_{i=1}^{t} T(z_i) \geq \alpha + f - \frac{f}{2} = \alpha + \frac{f}{2}$$

On the other hand, if we sample from $c_2$, the mean of $T(z)$ for $z \sim c_1$ is $\alpha$, so applying the Chernoff Bound, we have w.p. at least $1 - \exp(-tf^2/2)$

$$\frac{1}{t}\sum_{i=1}^{t} T(z_i) \leq \alpha + \frac{f}{2}$$

Using union bound, we've found a test such that w.p. at least $1 - 2\exp(-tf^2/2)$ it takes 1 under $c_1$ and 0 under $c_2$. Therefore the statistical difference between $c_1$ and $c_2$ is at least $1 - 2\exp(-tf^2/2)$. □

**Lemma 2** (The XOR reduction)**.** $SD^{c,f} \leq SD^{c^t, f^t}$

We construct the new distributions by mapping $(c_0, c_1)$ to

$$(D_0, D_1) = ((c_0, c_1)_0^{\bigoplus t}, (c_0, c_1)_1^{\bigoplus t})$$

We will define $((c_0, c_1)_0^{\bigoplus t}, (c_0, c_1)_1^{\bigoplus t})$ recursively below.

For pairs of random variables $(X_0, X_1)$ and $(y_0, y_1)$, we construct a new pair $(Z_0, Z_1)$ as follows:

$$Z_0 = \begin{cases} (X_0, y_0) \text{ wp } \frac{1}{2} \\ (X_1, y_1) \text{ wp } \frac{1}{2} \end{cases}$$

and

$$Z_1 = \begin{cases} (X_0, y_1) \text{ wp } \frac{1}{2} \\ (X_1, y_0) \text{ wp } \frac{1}{2} \end{cases}$$

Then $\forall\ \alpha, \beta$, we have

$$\begin{aligned} &P[Z_0 = (\alpha,\beta)] - P[Z_1 = (\alpha,\beta)] \\ &= \frac{1}{2}(P[X_0 = \alpha]P[y_0 = \beta] + P[X_1 = \alpha]P[y_1 = \beta] - (P[X_0 = \alpha]P[y_1 = \beta] + P[X_1 = \alpha]P[y_0 = \beta]) \\ &= \frac{1}{2}(P[X_0 = \alpha] - P[X_1 = \alpha])(P[y_0 = \beta] - P[y_1 = \beta]) \end{aligned}$$

Therefore, $\delta(Z_0, Z_1) = \delta(X_0, X_1)\delta(y_0, y_1)$.

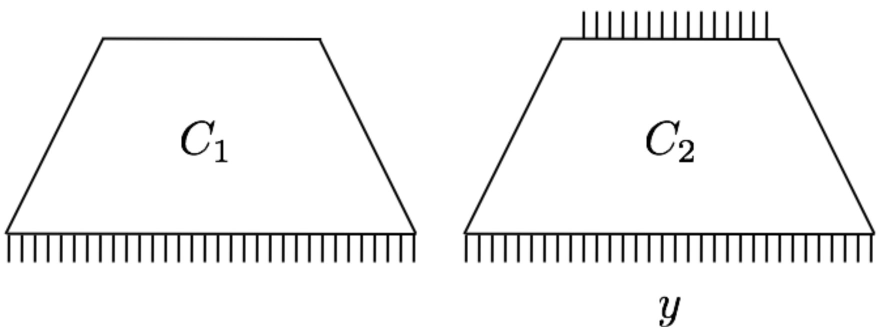
We define $((c_0, c_1)_0^{\bigoplus t}, (c_0, c_1)_1^{\bigoplus t})$ by recursively apply the above operations. $\delta(D_0, D_1) = \delta(C_0, C_1)^t$ follows by induction on $t$. Lemma 2 then follows.

To prove Theorem 1, we apply the direct product reduction and the XOR reduction multiple times.

(i) $SD^{\frac{1}{3},\frac{2}{3}} \to SD^{\frac{1}{n^2},\frac{1}{n^{0.8}}}$. We achieve this by using the XOR reduction with $t = O(\log n)$

(ii) $SD^{\frac{1}{n^2},\frac{1}{n^{0.8}}} \to SD^{\frac{1}{4},1-exp(-n^{0.4})}$. We achieve this by using the direct product reduction with $t = \frac{n^2}{4}$.

(iii) $SD^{\frac{1}{4},1-\exp(-n^{0.4})} \to SD^{\frac{1}{4n^{.1}},1-exp(-n^{0.3})}$. We achieve this by using the XOR reduction with $t = n^{.1}$.




# 4 Reduction of SD to its complement

In the last lecture, we mentioned that $\text{SD}^{c,f} \equiv \overline{\text{SD}}^{c,f}$, which means they are computationally equivalent, up to poly-time reductions. We only need to prove $SD \leq \overline{\text{SD}}$ since we can then apply this to $\overline{\text{SD}}$. The below proofs were originally proposed in [2] and was then presented in [1, 3].

**Theorem 2.** *$SD \leq \overline{SD}$, which means we can find a polytime reduction $(c_1, c_2) \to (D_1, D_2)$ such that*

$$(D_1, D_2) \textit{ are } \begin{cases} \textit{far if } (c_1, c_2) \textit{ are close} \\ \textit{close if } (c_1, c_2) \textit{ are far} \end{cases}$$

## 4.1 Entropy Difference

We consider the computational problem of entropy difference.

**Definition 3.** For distributions $(c_1, c_2)$, the entropy difference problem $\text{ED}^k$ is to decide whether $H(c_1) \geq H(c_2) + k$ (YES) or $H(c_2) \geq H(c_1) + k$ (NO).

We need to following properties to finish the proof of Theorem 2.

(i) $\mathrm{ED}^{\frac{1}{\sqrt{n}}} \leq \mathrm{ED}^{\sqrt{n}}$ (like before we repeat: $(c_0, c_1) \to (c_0^t, c_1^t)$)

(ii) $\mathrm{SD} \leq \mathrm{ED}$

(iii) $\mathrm{ED} \leq \overline{\mathrm{ED}}$ (the proof is trivial, we just map $(c_1, c_2) \to (c_2, c_1)$)

(iv) $\mathrm{ED} \leq \mathrm{SD}$ (we can convert it to $(\overline{\mathrm{ED}} \leq \overline{\mathrm{SD}})$)

To prove (ii) $\mathrm{SD} \leq \mathrm{ED}$, given $(c_0, c_1)$, we map it to $(D_0, D_1)$ such that

- $D_0 = (b, c_b(X))$ where $b$ and $X$ are picked at random
- $D_1 = (b', c_b(X))$ where $b$, $b'$ and $X$ are picked at random

We can see that $H(D_1) = H(c_b) + 1$. If $c_0$ and $c_1$ are very far apart, then we we can infer $b$ from $c_b(X)$, so $H(D_0)$ would be close to $H(c_b) + 1$ and be significantly smaller than $H(D_1)$. Otherwise, if $c_0$ and $c_1$ are really close, then $b$ cannot be inferred from $c_b(X)$ and $H(D_1) \approx H(D_0)$.

To prove (iii), we use "extractors" to manipulate random variables. If there is already sufficient entropy, we can transform random variable with entropy $k$ to uniform distribution on $k - o(1)$ bits. However, we cannot do this with insufficient entropy. To do this, there are three problems

1. Need Extractor + analysis (easy, pairwise independence)
2. This works for min entropy, but we have entropy. ("entropy flattening", "AEP", $C_1 \to C_1^t$)
3. Even if we have flat source, we don't know entropy of $c_1$ or $c_2$, so we have no idea of which hash function to use to extract entropy.

The key solution is proposed in [2]. Assume $C_2$ has more entropy than $C_1$. Consider a random output of $C_1$ and a random hash function $h$. We map $(C_1, C_2)$ to $((C_1(X), h, h(X, C_2(y)))$. Then we can prove that

$$H(X|C_1(X)) = m - H(C_1(X))$$

So $H(X, C_2(y)|C_1(X)) = m - H(C_1(X)) + H(C_2(y))$. Because the entropy of $X$ and $(X, C_2(y))$ differ (conditioning on $C_1(X)$), when we feed them into $h$, we will get distinguishable distributions (one should be much "more uniform" than the other). Notice we don't need to know how much entropy in $C_1$.

With the above properties, we can readily prove Theorem 2.

**Exercise 2.** *Prove Theorem 2 using the above proved properties.*

*Proof.*

$$\begin{aligned} \mathrm{SD} &\leq \mathrm{ED} \text{ (property (ii))} \\ &\leq \overline{\mathrm{ED}} \text{ (property (iii))} \\ &\leq \overline{\mathrm{SD}} \text{ (converted property (iv))} \end{aligned}$$

□

## References


[1] Goldreich, O., and Vadhan, S. P. On the complexity of computational problems regarding distributions (a survey). In *Electronic Colloquium on Computational Complexity (ECCC)* (2011), vol. 18, p. 4.

[2] Okamoto, T. On relationships between statistical zero-knowledge proofs. *Journal of Computer and System Sciences 60*, 1 (2000), 47–108.

[3] Vadhan, S. P. *A study of statistical zero-knowledge proofs*. PhD thesis, Massachusetts Institute of Technology, 1999.